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STUDY OF THE DYNAMICS OF ORBITAL ASSEMBLIES
INCLUDING INTERACTIONS WITH GEOMETRICAL
APPENDAGES

Contract Nos. NAS 8-26131
NAS 8-28169

UNIFIED FLEXIBLE SPACECRAFT SIMULATION PROGRAM
(UFSSP)

METHODOLOGY REPORT

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I

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I. Introduction

This document presents the complete equations for the Unified Flexible Spacecraft Simulation (UFSS) Program developed by TRW Systems for the NASA/MSFC under Contract NAS 8-26131. This general purpose simulation program is based on an algorithm which utilizes the digital computer to synthesize the dynamic and kinematic equations for a topological tree configuration of N interconnected bodies (the interconnected system of bodies forms no closed loops), the terminal members of which may be flexible. (For illustrative purposes, Figure 2.1 depicts such a spacecraft model (N=15) where the possible flexible bodies are shaded.) Necessary input quantities to the dynamics subroutine include the mass and inertia properties of each body and the flexible characteristics of each terminal member in addition to the specification, for each body, of those bodies to which it connects. This latter description involves the specification of the number of rotational degrees of freedom at each interconnection along with the associated position vectors defining these connections relative to the mass centers of the bodies involved. These position vectors can be input as time-varying functions if desired, thus affording the capability of studying the effects of time-varying hinge locations. Springs and dampers are assumed to act at each interconnection and structural damping in the flexible terminal members is included in the form of equivalent viscous damping.

Figure 1.1 presents the major subroutines of the UFSS program. The Dynamics Subroutine is the subject of Sections III, IV and V of this report. The Disturbance Subroutine is documented in Section VI; the Orbit Subroutine is documented in Section VII; the Control Interface Subroutine is documented in Section VIII.

[Note that although no control laws are implemented at this time, this interface routine provides for the future addition of any specific or generalized control subroutines.]

Finally, the Modal Subroutine is documeted in Section IX with a list of symbols given in Section X. Appendix A details the derivation of the dynamic equations, while Appendix B contains the derivation of the flexible disturbances.

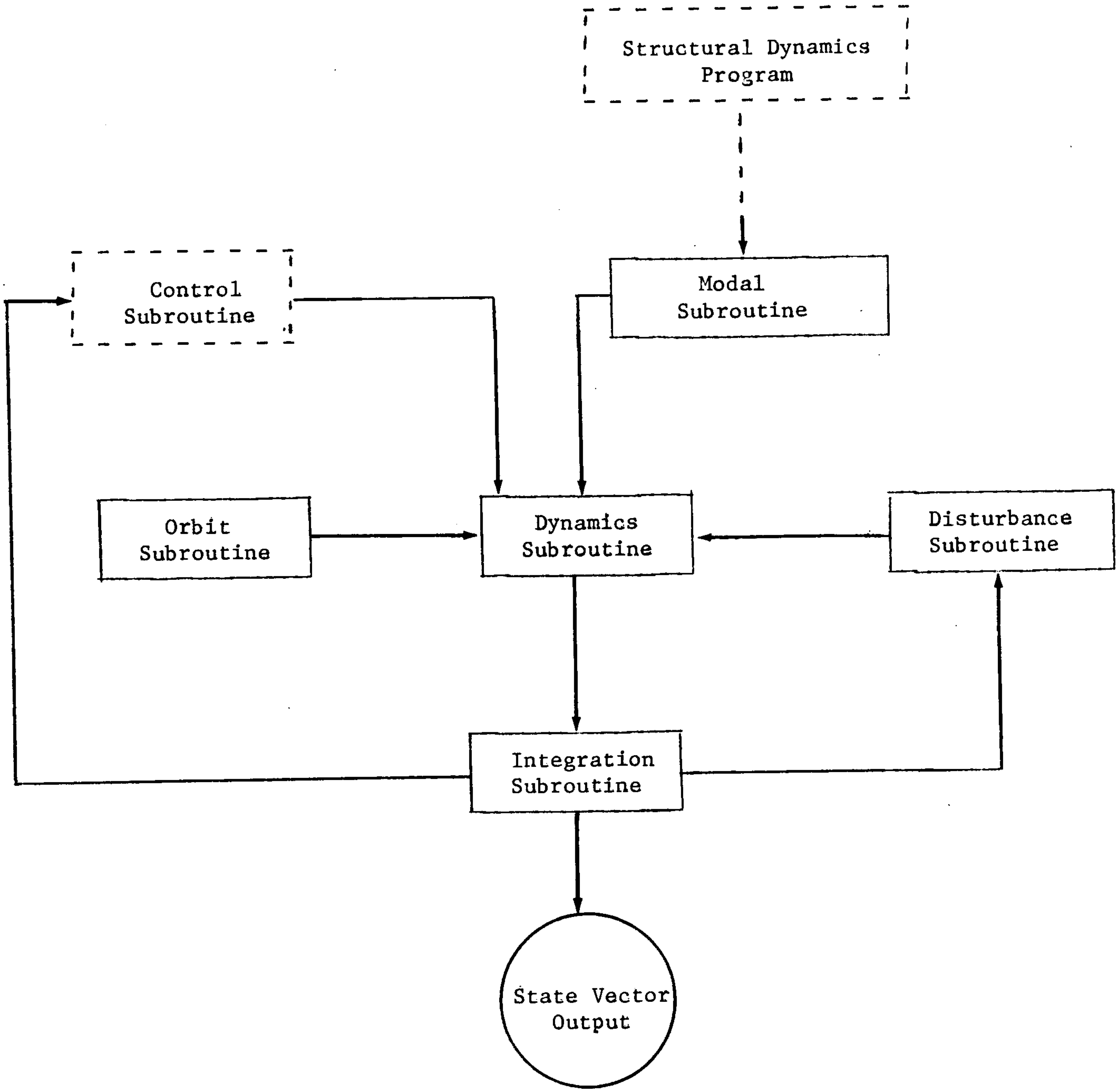


Figure 1.1 Schematic of the Major Subroutines of the UFSSP

II. Description of the System Model and Notation

Consider the case of a multibodied flexible spacecraft modeled as a system of N bodies interconnected in a topological tree configuration such that only the terminal bodies may be flexible. Figure 2.1 below exhibits such a configuration where N=15 and the possible flexible bodies are shaded.

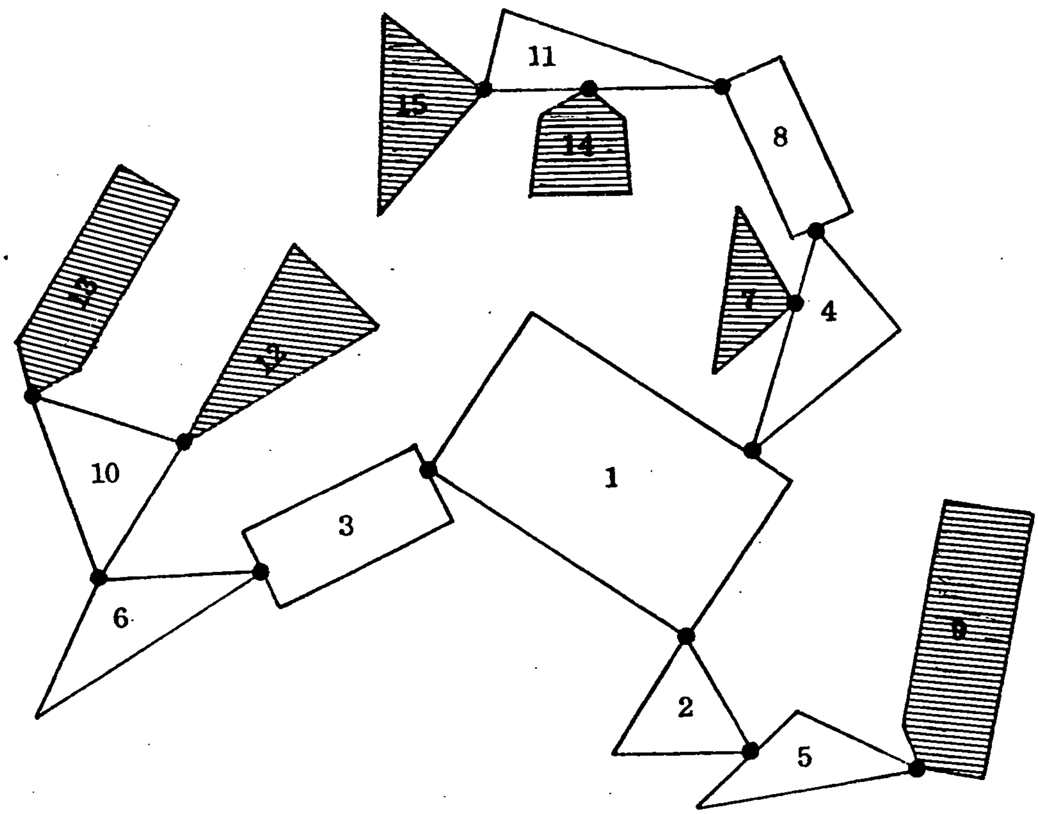


Figure 2.1 Spacecraft Model in a Topological Tree Configuration.

This figure also exemplifies the method by which the user numerically identifies the individual bodies.

2.1 Topological Tree Model Specification

In particular, whenever a physical system is modeled as in Figure 2.1, it is possible to identify a specific body whose attitude relative to some external reference coordinate set is of prime interest; this is usually that portion of the spacecraft housing the payload and this particular body will be denoted as Body 1 of the system. [An example of a possible external reference coordinate set for this program is a set located on a user defined Kepler orbit such as specified in Section VII.]

The remaining bodies are labeled in numerical sequence in a fashion which denotes the minimum number of interconnections that must be crossed in traversinga path from the specific body back to Body 1. The number of connecting points crossed is defined to be the level of the body. (Thus, there is a unique body having level zero, and this is Body 1.)

In general, if Body m is connected to Body n and the level of Body m is greater than that of Body n, then Body m is defined to be a branch of Body n and Body n is defined to be the limb of Body m. A sub-branch of Body n is any member of higher level than that of Body n for which the latter forms a link in the chain connecting the member to Body 1.

The rule for numbering the individual bodies in the topological tree configuration is simply that:

$m \geq n$ implies that the level of Body $m \geq$ the level of Body n

Figure 2.1 clearly illustrates this labeling scheme. There are three level one bodies (2, 3, 4), four level two bodies (5, 6, 7, 8), three level three bodies (9, 10, 11) and four level four bodies (12, 13, 14, 15).

From Figure 2.1, the limb-branch relationships given in Table 2.1 are determined. These relationships, along with the specification of each body as flexible or rigid plus the rigid-body and flexible degrees of freedom for each body, constitute input to the computer program.

Table 2.1. Configurational Input Relationships for Figure 2.1

	Branch	Limb	Type of Body	Rigid-Body Degrees of Freedom (p_j)	Flexible Degrees of Freedom (n_j)
Body	1	0	R	6	0
	2	1	R	2	0
	3	1	R	1	0
	4	1	R	3	0
	5	2	R	3	0
	6	3	F	0	3
	7	4	R	1	0
	8	4	R	2	0
	9	5	R	1	0
	10	6	F	3	5
	11	8	R	3	0
	12	10	F	1	4
	13	10	F	2	0
	14	11	F	1	4
	15	11	F	3	4

It should be noted in Table 2.1 that Body 13 is entered as a flexible body with zero flexible degrees of freedom. This case is allowed to cover the option of constraining all flexible degrees of freedom for a given flexible body for preliminary analyses.

2.2 Definition of Coordinate Systems and Basic Vector Quantities

Figure 2.2 depicts a multibodied flexible spacecraft system modeled as a topological tree configuration of bodies traveling through space in the vicinity of an attracting body. In most cases, motion of the configuration is expressed with respect to an orbital reference axis frame although this is not mandatory and the reference axis frame may be arbitrarily specified if so desired.

Several right-handed, orthogonal coordinate frames are used extensively in the following development. The pertinent axis sets are described below, the inertial reference frame being given with respect to the earth as the orbited body.

x_ε^e ; $\underline{e}_\varepsilon^e$ $(\varepsilon=1,2,3) \rightarrow$ Coordinates and unit vectors of the inertially fixed coordinate system with origin 0_e at the earth's center. $\underline{e}_1^e$ and $\underline{e}_3^e$ lie in the equatorial plane with $\underline{e}_2^e$ normal to this plane and pointing northward; $\underline{e}_3^e$ is directed along the autumnal equinox.

x_α^r ; $\underline{e}_\alpha^r$ $(\alpha=1,2,3) \rightarrow$ Coordinates and unit vectors of the moving reference coordinate frame. If the reference frame is an orbital one(Section VIII), then its origin 0_r lies on the user defined Kepler orbit with $\underline{e}_1^r$ and $\underline{e}_3^r$ lying in the orbit plane and $\underline{e}_2^r$ normal to it. $\underline{e}_3^r$ points toward the earth's center and $\underline{e}_1^r$ forms an acute angle with the orbital tangential velocity vector.

$\bar{R}^r = R_\varepsilon^r \ \underline{e}_\varepsilon^e$ $\longrightarrow$ Position vector of the reference coordinate center 0_r with respect to the earth's center 0_e .

$\bar{R}^1 = R_\beta^1 \ \underline{e}_\beta^r$ $\longrightarrow$ Position vector of Body 1 mass center 0_1 with respect to the reference coordinate center 0_r .

[Note: repeated subscripts here and throughout this document denote summation over the range of the repeated subscript; i.e., $R_\beta^1 \ \underline{e}_\beta^r = R_1^1 \ \underline{e}_1^r + R_2^1 \ \underline{e}_2^r + R_3^1 \ \underline{e}_3^r$]

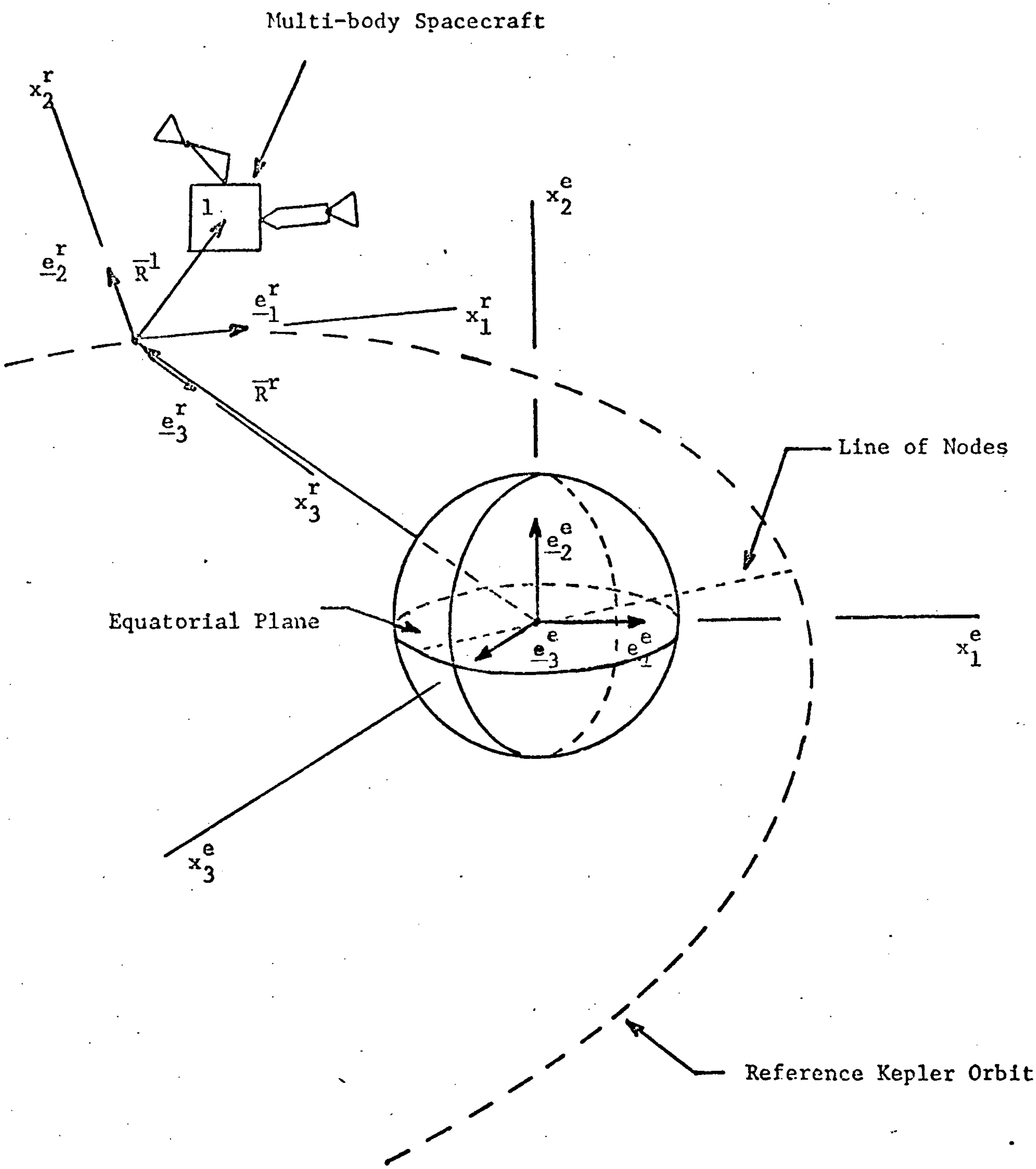


Figure 2.2. Details of Inertial and Orbital Reference Axis Frames

In what follows, vector quantities will be expressed in tensor (or indical notation) form simply by denoting the components. Thus

$$\bar{R}^r \longrightarrow R^r_\alpha \quad .$$

In addition, an "outer product" matrix is formed from a given vector as follows:

$$\tilde{\omega}^j_{\alpha\beta} = \begin{bmatrix} 0 & -\omega^j_3 & \omega^j_2 \\ \omega^j_3 & 0 & -\omega^j_1 \\ -\omega^j_2 & \omega^j_1 & 0 \end{bmatrix}$$

so that, transforming from vector to index notation,

$$\bar{f} \times \bar{g} = \tilde{f}_{\alpha\beta}\, g_\beta \quad .$$

Figure 2.3 presents a schematic of a terminal flexible body (Body j) and its limb (Body i). The following coordinate frames and vector quantities can now be defined:

x^k_α ; $\underline{e}^k_\alpha$ $(\alpha=1,2,3)$ $\rightarrow$ Coordinates and unit vectors of an axis frame fixed to Body k. If Body k is rigid, then the origin of this frame 0_k is located at its mass center. If Body k is a terminal flexible member, then 0_k is located at the connecting point of Body k with its limb.

$x^{k_o}_\beta$; $\underline{e}^{k_o}_\beta$ $(\beta=1,2,3)$ $\rightarrow$ Coordinates and unit vectors of an axis frame with fixed orientation relative to the limb of Body k. The origin of this frame is coincident with 0_k. Normally this frame is used to define some nominal orientation of Body k relative to its limb with the attitude variables of Body k defining the orientation of the x^k_α relative to the $x^{k_o}_\beta$.

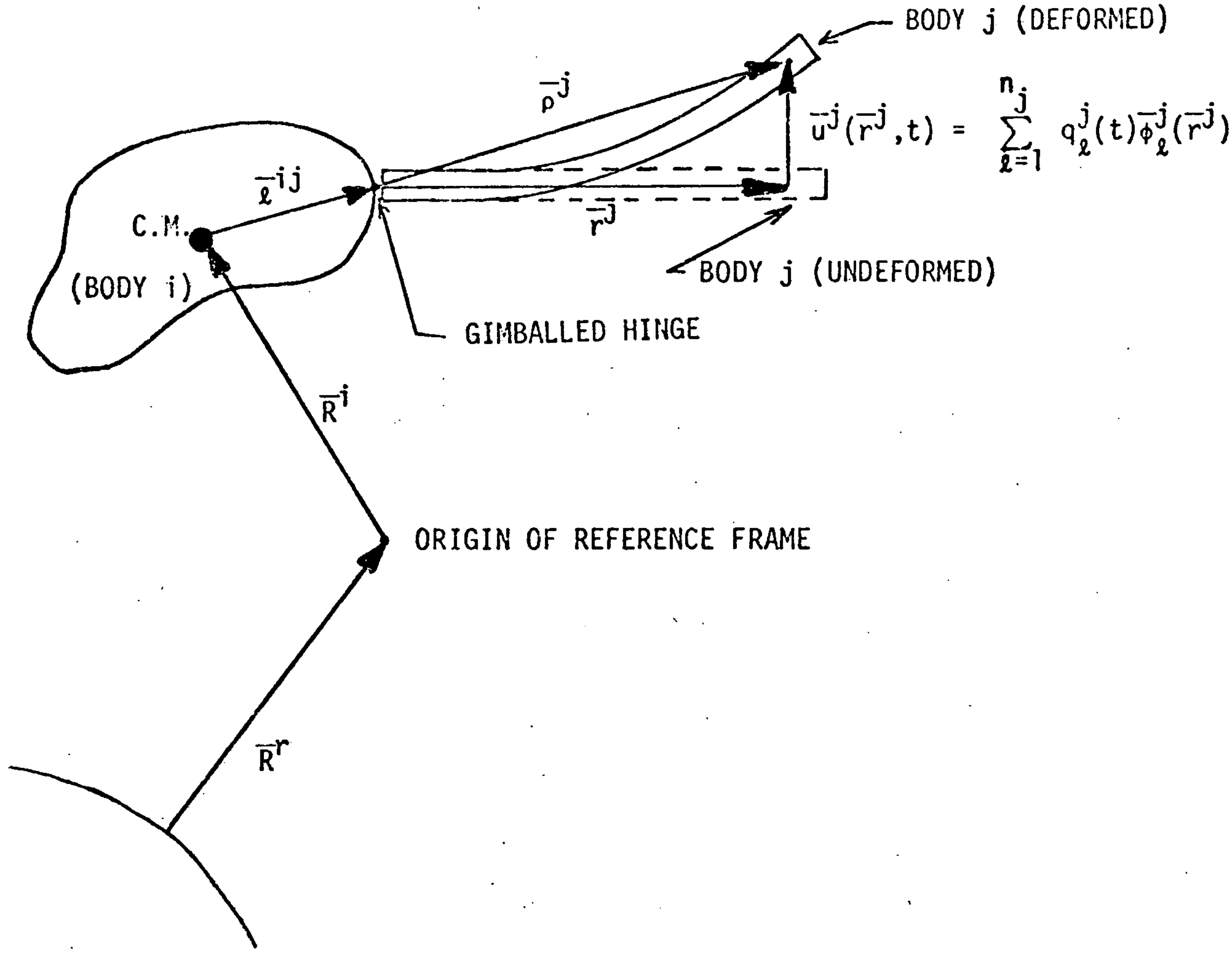


Figure 2.3. Two-Body Schematic

$\bar{\omega}^r = \omega^r_\alpha \underline{e}^r_\alpha \longrightarrow$ Angular velocity vector of the reference frame.

$\bar{\omega}^k = \omega^k_\alpha \underline{e}^k_\alpha \longrightarrow$ Angular velocity vector of the x^k_α frame

$\bar{R}^k = R^k_\alpha \underline{e}^r_\alpha \longrightarrow$ Position vector of 0_k with respect to 0_r .

$\bar{\ell}^{ij} = \ell^{ij}_\alpha \underline{e}^i_\alpha \longrightarrow$ Position vector from the Body i mass center to its connection with Body j.

$\bar{r}^j = r^j_\alpha \underline{e}^j_\alpha \longrightarrow$ For Body j a flexible body, r^j is the position vector from the connecting point to an arbitrary mass point in Body j when the body is undeformed.

The various coordinate frames are related through direction cosine matrices as follows:

$$x^j_\alpha = A^{jjo}_{\alpha\beta} \; x^{jo}_\beta$$

$$x^j_\alpha = A^{ji}_{\alpha\beta} \; x^i_\beta$$

$$x^j_\alpha = A^{jr}_{\alpha\beta} \; x^r_\beta$$

where

$$A^{jjo}_{\alpha\beta} = \underline{e}^j_\alpha \cdot \underline{e}^{jo}_\beta$$

$$A^{ji}_{\alpha\beta} = \underline{e}^j_\alpha \cdot \underline{e}^i_\beta$$

$$A^{jr}_{\alpha\beta} = \underline{e}^j_\alpha \cdot \underline{e}^r_\beta$$

All adjacent bodies are assumed to be connected through a gimbal hinge (see Section IV.1) with gimbal angular rotations expressed by the coordinates

θ_α^j $(\alpha=1,2,3)$ = the gimbal rotations defining the orientation of the x_α^j frame with respect to the $x_\beta^{j_o}$ frame.

In addition, once bodies have been combined, the following column matrix is used in the sequel:

$\hat{\ddot{\theta}}_b^{ij}$ for Body j a branch of Body i, this column matrix has a rotational component for every flexible and rotational degree of freedom of Body i's branches and sub-branches numbered $\geq j$

When a terminal body, say Body j, is deformed, the position vector $\bar{\rho}^j = \rho_\alpha^j \, \underline{e}_\alpha^j$ of an arbitrary mass point in the body is given by (see Figure 2.3)

$$\bar{\rho}^j\,(\bar{r}^j,\, t) = \bar{r}^j + \bar{u}^j\,(\bar{r}^j,\, t)$$

where $\bar{r}^j$ is the position vector of the mass point when Body j is undeformed. In this program, $\bar{u}^j$ is assumed to be representable as a finite sum of vector field functions (orthogonal functions) with time-varying coefficients:

$$\begin{aligned}\bar{u}^j\,(\bar{r}^j,\, t) &= \sum_{\alpha=1}^{3} u_\alpha^j \, \underline{e}_\alpha^j \\ &= \sum_{\ell=1}^{\eta_j} q_\ell^j\,(t)\;\bar{\phi}_\ell^j\,(\bar{r}^j) \\ &= \sum_{\ell=1}^{\eta_j} q_\ell^j\,(t) \sum_{\alpha=1}^{3} \phi_{\ell\alpha}^j \, \underline{e}_\alpha^j\end{aligned}$$

where

$\bar{\phi}_\ell^j\,(\bar{r}^j)$ = the ℓth orthogonal function describing spatial variation of $\bar{u}^j$.

$q_\ell^j\,(t)$ = generalized coordinate describing the time variation of $\bar{u}^j$.

η_j = number of terms in the series expansion.

III. Synthesis of the Dynamic Equations

Attention will now be focused upon the detailed description of the algorithm utilized by the computer to generate the system dynamic and kinematic equations. The state vector is taken to consist of the scalar elements

$$A^{ir}_{\alpha\beta}\ ,\ \omega^{1}_{\alpha}\ ,\ R^{1}_{\alpha}\ ,\ \dot{R}^{1}_{\alpha}\ ,\ \theta^{j}_{\gamma}\ ,\ \dot{\theta}^{j}_{\gamma}\ ,\ q^{j}_{k}\ ,\ \dot{q}^{j}_{k}$$

where

$$\alpha, \beta = 1,2,3$$
$$j = \ ,\ldots,N$$
$$k = 0,1,\ldots,n_j$$
$$\gamma = 0,1,\ldots,p_j$$

with n_j being the number of flexible degrees of freedom of Body j and p_j being the number of rigid-body degrees of freedom of Body j with respect to its limb (n_j and p_j may be zero).

The inductive algorithm used to synthesize the dynamic equations for a spacecraft modeled as in Figure 2.1 is based upon an operation which, given the dynamic equations for two separate systems, generates the dynamic equations when the two systems are coupled together. Specifically, suppose the dynamic equations are known for both Systems A and B in Figure 3.1, and that System C is a combination of these two through an r $(0 \le r \le 3)$ degree of rotational freedom interconnection. (Note that only the shaded bodies in Systems A, B and C may be flexible.)

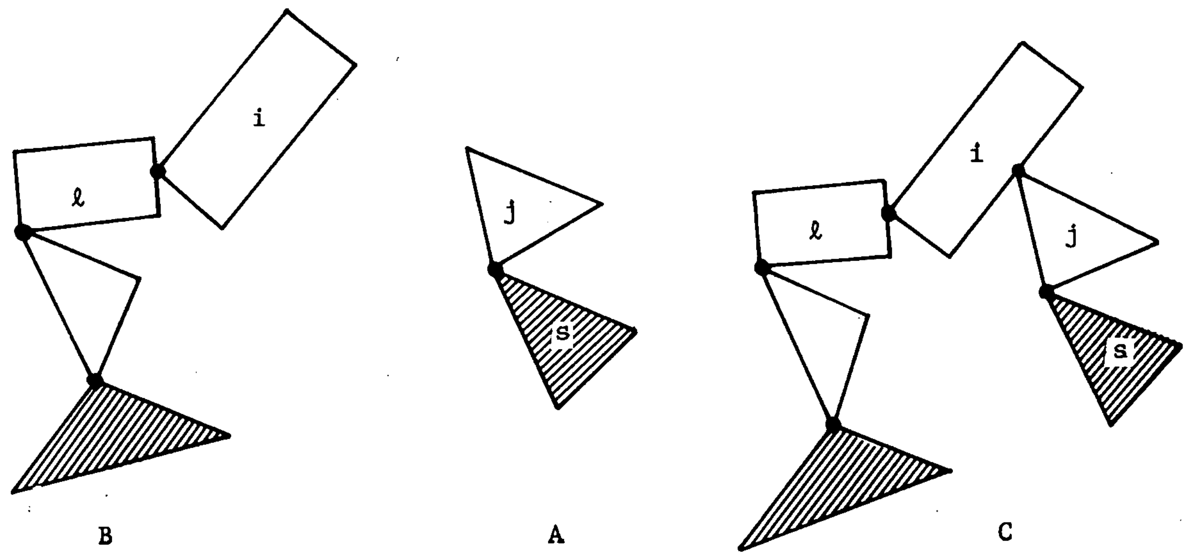


Figure 3.1. Combining Two Systems to Form a Third

The algorithm providing the dynamic equations for the combined System C is termed the <u>Combining Unit</u> and can be represented schematically by a two-input — one-output device as pictured in Figure 3.2.

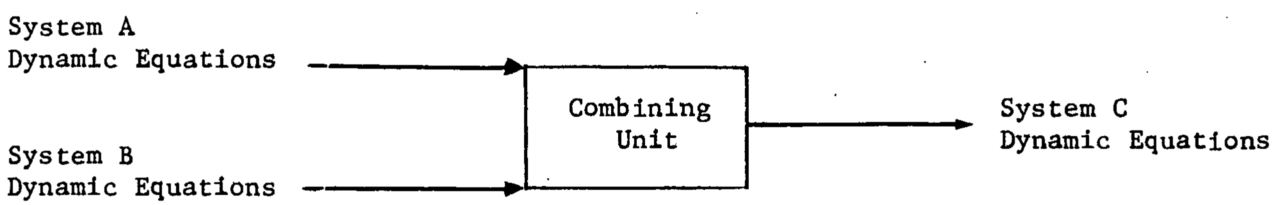


Figure 3.2. Schematic of the Combining Unit

The actual combining operation is accomplished by eliminating from the dynamical equations the forces and torques of constraint between the Systems A and B. When applied to an actual spacecraft modeled as in Figure 2.1, this Combining Unit is utilized repetitively to synthesize the dynamic equations through appropriate interpretations of the Systems A, B and C.

Thus, there are two elementary components in the synthesizing algorithm: first, the Combining Unit and second, the Sequencing Algorithm which specifies the appropriate Systems A, B and C at each application of the Combining Unit.

Let us first consider the latter component.

3.1 The Sequencing Algorithm

In order to specify the sequence of combining operations, one need merely list for each branch its corresponding limb and the number of interconnecting rotational degrees of freedom. Once this information is supplied, a computer algorithm determines the sequence of combining operations necessary to synthesize the system equations.

For example, consider the configuration shown in Figure 2.1. From the input information supplied in Table 2.1, the Sequencing Algorithm determines the step by step procedure outlined in Table 3.1 below, defining the Systems A, B and C at every application of the Combining Unit.

Assuming that the lowest level body in System A has a level greater than that of the lowest level body in System B, the development initiates at the highest numbered body (15) and considers the interconnection with its limb (11).

---

*see Section III.2

Thus, upon the first pass through the Combining Unit, the generated equations of motion (C) apply to the two-body system 15-11, where the constraint forces and torques between these two bodies have been eliminated and the torques along degree of freedom axes are specified by other routines of the simulation, such as the control system subroutine. In a similar way, the second pass combines body 14 with the system 15-11 to form the three body result 15-14-11. This technique is now successively repeated until, after the 14th pass, the complete equations of motion (i.e., the second derivative of the state vector components) reside in C and are ready for the computer's integration package. This sequence of operations is retained by the computer and applied, at each integration step, to form the dynamic equations.

Table 3.1. Synthesis of Dynamic Equations for System of Figure 2.1

Use No.	System A	System B	System C	Input Parameters Required for Body
1	15	11	15-11	15,11
2	14	C	15-14-11	14
3	C	8	15-14-11-8	8
4	C	4	15-14-11-8-4	4
5	7	C	15-14-11-8-7-4	7
6	C	1	15-14-11-8-7-4-1	1
			Store C in C1	- - -
7	13	10	13-10	13,10
8	12	C	13-12-10	12
9	C	6	13-12-10-6	6
10	C	3	13-12-10-6-3	3
11	C	C1	13-12-10-6-3-15-14-11-8-7-4-1	- - -
			Store C in C1	
12	9	5	9-5	9,5
13	C	2	9-5-2	2
14	C	C1	15 through 1	- - -
	C now houses the system dynamic equations			

Determination of the new translational and rotational positions is a considerably simpler task as it is merely necessary to integrate directly the rate variables already present in the state vector. This procedure applies to all but the body 1 rotations for which the attitude direction cosines are desired. These can be obtained by integrating the conventional direction cosine equations as will be demonstrated later. [see (4-52)]

When defining the Combining Unit in the following section, it is convenient to identify a system by specifying the lowest leveled body and its lowest numbered branch. As shown in Figure 3.3, let Body j of Level (N+1) be the lowest leveled body of System A [here, N is arbitrary and does not refer to the total number of bodies in the configuration] and let Body s of Level (N+2) be the lowest numbered branch of Body j. Let Body i of Level N be the lowest leveled body of System B and Body ℓ of Level (N+1) be the lowest numbered branch of Body i. [Note that because the sequencing algorithm commences with the highest numbered bodies, it is true that $j<\ell$.] Following the combining process, Body j becomes the lowest numbered branch of Body i and the entire combined system is designated System C. Thus, system identification is as follows:

$$\text{System A} \longrightarrow \text{System } \widehat{js}$$

$$\text{System B} \longrightarrow \text{System } \widehat{i\ell}$$

$$\text{System C} \longrightarrow \text{System } \widehat{ij}$$

3.2 The Combining Unit

As exemplified in Table 3.1, the Combining Unit is utilized repetitively to synthesize the dynamic equations for a combined System C given the dynamic equations for its components, Systems A and B. In this program, the dynamic equations for the rigid bodies of the system are the standard Euler equations while the dynamic equations for the terminal flexible bodies are obtained via a Lagrangian approach. Because of dissimilarities in the forms of the equations for the flexible and rigid bodies, two distinct combining

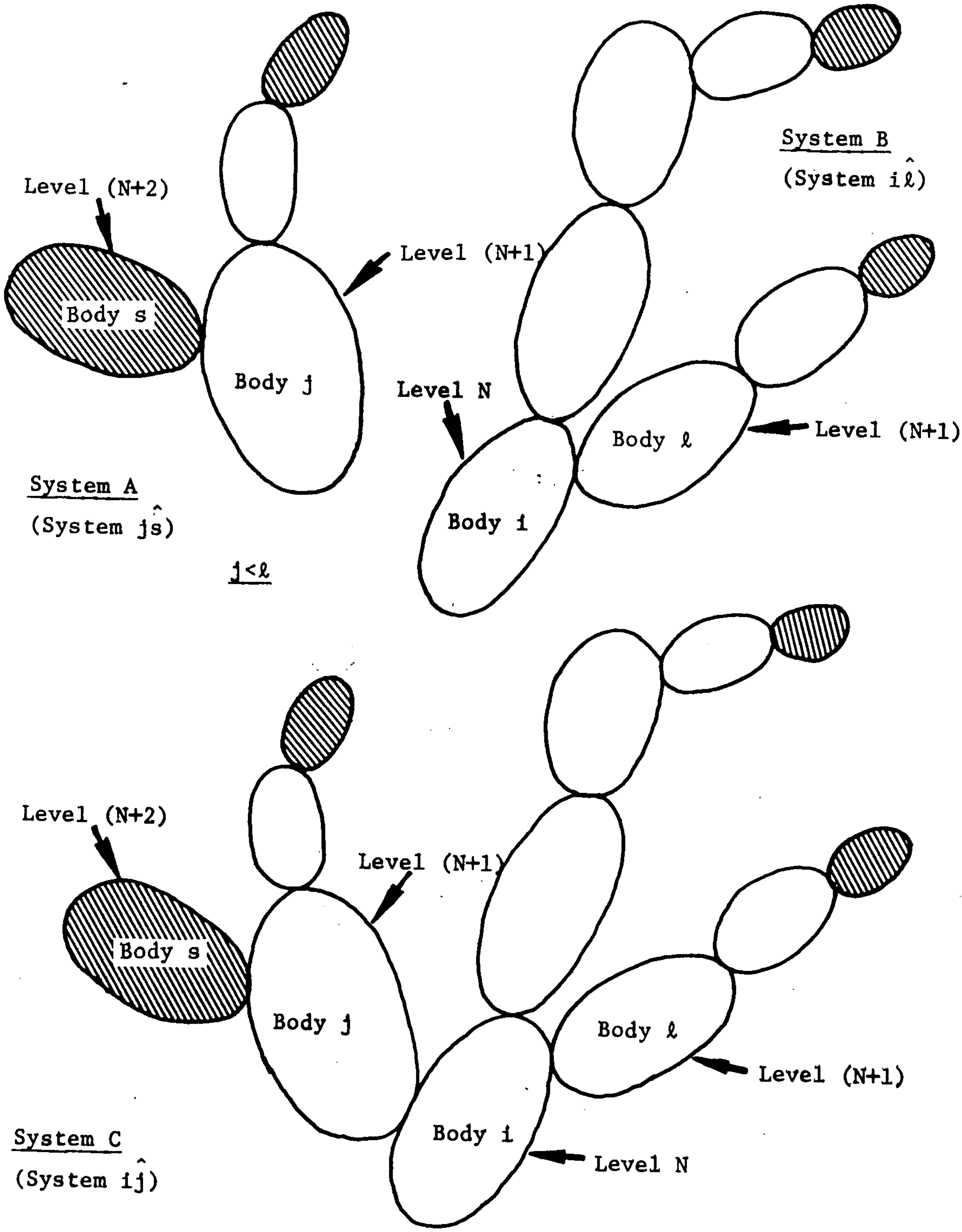


Figure 3.3. The Combining Process for Two Systems of Bodies

algorithms are utilized in the Combining Unit. Choice of the proper combining algorithm depends on whether or not System A is a single flexible body in a given combining operation (see Figure 3.4 where the Combining Unit is represented by the blocks contained within the dashed lines).

The following section contains details of the Rigid Combining Algorithm used when System A is not a single flexible body; Section V defines the Flexible Combining Algorithm used when System A is a single flexible body. Derivations of the governing equations are contained in Appendix C.

IV. Rigid Combining Algorithm Specification*

Assume that System A in a given combining operation (Use No. in Table 3.1) is not a single flexible body. In this case, the Rigid Combining Algorithm is utilized to synthesize the equations for System C given those for Systems A and B. Prior to the first pass through this algorithm, it is necessary to compute certain auxiliary variables that are not elements of the dynamic state vector, but that can be algebraically determined from this vector. [In addition to the auxiliary variables specified in the following section, the gimbal torques and certain quantities associated with the flexible bodies should be computed at the same time. These quantities are specified in Sections 4.4, 5.1 and 5.3.]

4.1 The Auxiliary Variables

Specifically, it is desirable to compute and store for all bodies j the scalar components of $A^{ir}_{\alpha\beta}$, R^{j}_{α} , $\dot{R}^{j}_{\alpha}$ and ω^{j}_{α} .

Using the notation of Section 2.2 and supposing Body i to be the limb of Body j, it follows that

$$A^{ji}_{\alpha\beta} = A^{jj_o}_{\alpha\gamma} \; A^{j_o i}_{\gamma\beta} \tag{4.1}$$

where $A^{j_o i}_{\gamma\beta}$ is the input transformation from the Body i axes to the nominal Body j axes. Let us first consider the transformation matrix $A^{jj_o}_{\alpha\beta}$. $A^{jj_o}_{\alpha\beta}$ can be obtained by three successive Euler rotations θ^{j}_{α} (α=1,2,3), satisfying the constraints of a gimballed hinge, such that

*this algorithm originally developed in Reference 3.

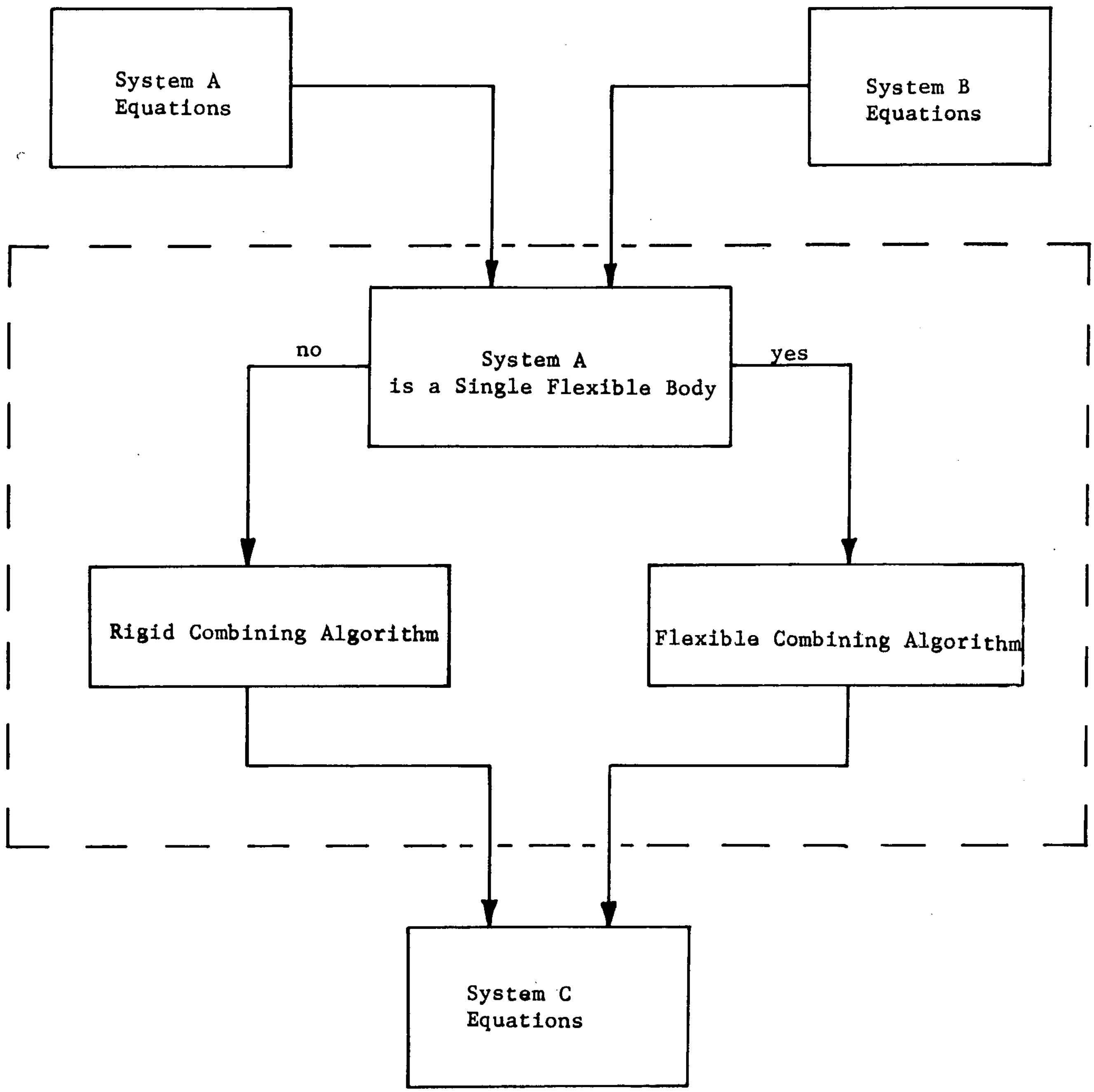


Figure 3.4. Elements of the Combining Unit

$$A^{jj_o}_{\alpha\beta} = G^{j3}_{\alpha\gamma}\ G^{j2}_{\gamma\delta}\ G^{j1}_{\delta\beta} \tag{4-2}$$

where

$$G^{j1}_{\delta\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^j_1 & \sin\theta^j_1 \\ 0 & -\sin\theta^j_1 & \cos\theta^j_1 \end{bmatrix} \tag{4-3}$$

$$G^{j2}_{\gamma\delta} = \begin{bmatrix} \cos\theta^j_2 & 0 & -\sin\theta^j_2 \\ 0 & 1 & 0 \\ \sin\theta^j_2 & 0 & \cos\theta^j_2 \end{bmatrix} \tag{4-4}$$

$$G^{j3}_{\alpha\gamma} = \begin{bmatrix} \cos\theta^j_3 & \sin\theta^j_3 & 0 \\ -\sin\theta^j_3 & \cos\theta^j_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4-5}$$

Rotational constraints in Equations (4-3) to (4-5) are handled by setting the pertinent θ^j_α identically equal to zero, with the condition that first θ^j_3 is constrained, then θ^j_2 and finally, θ^j_1 if necessary.

In a similar fashion,

$$A^{j_o i}_{\alpha\beta} = G^{j_o 3}_{\alpha\gamma}\ G^{j_o 2}_{\gamma\delta}\ G^{j_o 1}_{\delta\beta} \tag{4-6}$$

where the $G^{jo}_{\delta\gamma}$ are identical to the $G^{j}_{\delta\gamma}$ defined by Equations (4-3) to (4-5) with θ^{j}_{α} replaced by θ^{jo}_{α} (α=1,2,3). The θ^{jo}_{α} are input parameters. In most instances the θ^{jo}_{α} will be constants; only when generalized relative displacement between Body j and Body i is desired will it be necessary to input the θ^{jo}_{α} as time-dependent quantities and in these instances the θ^{jo}_{α} (t) and $\bar{\ell}^{ij}$ (t) must be consistently prescribed.

Angular velocities are determined through the relation

$$\omega^{j}_{\alpha} = A^{jjo}_{\alpha\beta} \; \omega^{jo}_{\beta} + G^{j+}_{\alpha\beta} \; \dot{\theta}^{j\circ}_{\beta} \tag{4-7}$$ *

where

$$G^{j}_{\alpha\beta} = \begin{bmatrix} (\cos\theta^{j}_{3} \;\; \cos\theta^{j}_{2}) & (\sin\theta^{j}_{3}) & 0 \\ (-\sin\theta^{j}_{3} \;\; \cos\theta^{j}_{2}) & (\cos\theta^{j}_{3}) & 0 \\ (\sin\theta^{j}_{2}) & 0 & 1 \end{bmatrix} \tag{4-8}$$

and

$G^{j+}_{\alpha\beta}$ = $G^{j}_{\alpha\beta}$ with those columns removed which imply no degree of freedom; i.e., if $\theta^{j}_{\alpha} \equiv 0$, then the αth column of $G^{j}_{\delta\gamma}$ is deleted in forming $G^{j+}_{\delta\gamma}$, (4-8a)

while $\dot{\theta}^{j\circ}_{\beta}$ is equal to $\dot{\theta}^{j}_{\beta}$ with those rows deleted which imply no degree of freedom. Thus, if θ^{j}_{β} has two degrees of freedom, then ω^{j}_{α} is given by

$$\omega^{j}_{\alpha} = A^{jjo}_{\alpha\beta} \; \omega^{jo}_{\beta} + \begin{bmatrix} (\cos\theta^{j}_{3} \;\; \cos\theta^{j}_{2}) & (\sin\theta^{j}_{3}) \\ (-\sin\theta^{j}_{3} \;\; \cos\theta^{j}_{2}) & (\cos\theta^{j}_{3}) \\ (\sin\theta^{j}_{2}) & (0) \end{bmatrix} \begin{Bmatrix} \dot{\theta}^{j}_{1} \\ \dot{\theta}^{j}_{2} \end{Bmatrix}$$

---

*(4-7) is not coded. The proper expression is (4-10).

In case the θ_α^{jo} (α=1,2,3) are constant, Equation (4-7) becomes

$$\omega_\alpha^j = A_{\alpha\beta}^{ji} \omega_\beta^i + G_{\alpha\beta}^{j+} \dot{\theta}_\beta^{j^o} . \tag{4-9}$$

In the general case where the θ_α^{jo} are time-dependent input functions, Equation (4-7) becomes

$$\omega_\alpha^j = A_{\alpha\beta}^{ji} \omega_\beta^i + G_{\alpha\beta}^{j+} \dot{\theta}_\beta^{j^o} + A_{\alpha\gamma}^{jjo} G_{\gamma\beta}^{jo} \dot{\theta}_\beta^{jo} \tag{4-10}$$

where $G_{\alpha\beta}^{jo}$ is defined similarly to $G_{\alpha\beta}^j$ with θ_α^{jo} replacing θ_α^j. (There is no need to define a $G_{\alpha\beta}^{jo+}$ since it is sufficient to set $\dot{\theta}_\alpha^{jo} = 0$ for any component θ_α^{jo} which is constant.)

It is evident that $A_{\alpha\beta}^{ji}$ can be computed directly from the state vector and the input parameters θ_α^{jo} (α=1,2,3). Adopting a procedure whereby the auxiliary variables associated with bodies of lower level are computed first, it can be assumed that $A_{\alpha\beta}^{ir}$ is already known. $A_{\alpha\beta}^{jr}$ then follows simply from the relation

$$A_{\alpha\beta}^{jr} = A_{\alpha\gamma}^{ji} A_{\gamma\beta}^{ir} . \tag{4-11}$$

To complete the description of the auxiliary variables,

$$A_{\alpha\beta}^{je} = A_{\alpha\gamma}^{jr} A_{\gamma\beta}^{re} \tag{4-11.1}$$

$$R_\alpha^j = R_\alpha^i + A_{\alpha\beta}^{ir(T)} \ell_\beta^{ij} - A_{\alpha\beta}^{jr(T)} \ell_\beta^{ji} \tag{4-12}$$

$$\begin{aligned} \dot{R}_\alpha^j = \dot{R}_\alpha^i &+ A_{\alpha\beta}^{ir(T)} \dot{\ell}_\beta^{ij} - A_{\alpha\beta}^{jr(T)} \dot{\ell}_\beta^{ji} \\ &+ \left\{ A_{\alpha\beta}^{ir(T)} \tilde{\omega}_{\beta\gamma}^i - \tilde{\omega}_{\alpha\beta}^r A_{\beta\gamma}^{ir(T)} \right\} \ell_\gamma^{ij} \\ &- \left\{ A_{\alpha\beta}^{jr(T)} \tilde{\omega}_{\beta\gamma}^j - \tilde{\omega}_{\alpha\beta}^r A_{\beta\gamma}^{jr(T)} \right\} \ell_\gamma^{ji} \end{aligned} \tag{4-13}$$

where the super-dot above a variable denotes time differentiation in the designated axis frame. In the above fashion, the auxiliary dynamic variables for all bodies can be obtained. It is merely necessary to consider them in the proper numerical sequence beginning with Body 1.

4.2 The Rigid Combining Algorithm

Suppose that System A has Body j as its member of lowest level, System B has Body i as its member of lowest level (i<j) and that A is to be connected to B to yield C as in Figure 3.3. Referring to this figure, it is also assumed that Body ℓ is the lowest numbered branch of Body i in System B, s is the lowest numbered branch of Body j, and $j<\ell$.

The dynamic equations for System A are as follows:

System A

$$\begin{bmatrix} B^{A11}_{k\ell} & B^{A12}_{k\beta} & B^{A13}_{k\beta} \\ B^{A21}_{\alpha\ell} & B^{A22}_{\alpha\beta} & B^{A23}_{\alpha\beta} \\ B^{A31}_{\alpha\ell} & B^{A32}_{\alpha\beta} & B^{A33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{\hat{js}}_{\ell} \\ \dot{\omega}^{j}_{\beta} \\ \ddot{R}^{j}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{A1}_{k} \\ C^{A2}_{\alpha} \\ C^{A3}_{\alpha} \end{bmatrix} \tag{4-14}$$

Here, $B^{A22}_{\alpha\beta}$, $B^{A23}_{\alpha\beta}$, $B^{A32}_{\alpha\beta}$ and $B^{A33}_{\alpha\beta}$ are 3 x 3 matrices with the indices α and β running from 1 to 3; C^{A2}_{α} and C^{A3}_{α} are 3 x 1 matrices; if $\ddot{\theta}^{\hat{js}}_{\ell}$ has M components, then $B^{A11}_{k\ell}$ is M x M, $B^{A12}_{k\beta}$ and $B^{A13}_{k\beta}$ are M x 3, $B^{A21}_{\alpha\ell}$ and $B^{A31}_{\alpha\ell}$ are 3 x M while C^{A1}_{k} is M x 1.

System B has a similar description and interpretation.

System B

$$\begin{bmatrix} B^{B11}_{mn} & B^{B12}_{m\beta} & B^{B13}_{m\beta} \\ B^{B21}_{\alpha n} & B^{B22}_{\alpha\beta} & B^{B23}_{\alpha\beta} \\ B^{B31}_{\alpha n} & B^{B32}_{\alpha\beta} & B^{B33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{\hat{i\ell}}_{n} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{B1}_{m} \\ C^{B2}_{\alpha} \\ C^{B3}_{\alpha} \end{bmatrix} \tag{4-15}$$

Finally, the output of the Rigid Combining Algorithm has a similar appearance.

System C

$$\begin{bmatrix} B^{C11}_{pq} & B^{C12}_{p\beta} & B^{C13}_{p\beta} \\ B^{C21}_{\alpha q} & B^{C22}_{\alpha\beta} & B^{C23}_{\alpha\beta} \\ B^{C31}_{\alpha q} & B^{C32}_{\alpha\beta} & B^{C33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{j}}_{q} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{C1}_{p} \\ C^{C2}_{\alpha} \\ C^{C3}_{\alpha} \end{bmatrix} \tag{4-16}$$

where

$$\ddot{\theta}^{i\hat{j}}_{q} = \begin{Bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \ddot{\theta}^{j\hat{s}}_{\ell} \\ \ddot{\theta}^{j}_{\gamma} \end{Bmatrix} \tag{4-17}$$

with the column matrix $\ddot{\theta}^{j}_{\gamma}$ including only the degree-of-freedom components at the interconnection of Bodies i and j.

To complete the description of the Rigid Combining Algorithm, it is now necessary only to explicitly specify the elements of the matrices B^{C} and C^{C} and to show how this inductive process can be initiated at the bodies of highest level by direct use of the computer input data.

$$B^{C11}_{pq} = \begin{bmatrix} B^{B11}_{mn} & 0 & 0 \\ 0 & B^{A11}_{k\ell} & P^{1}_{k\gamma} \\ 0 & P^{2}_{\alpha\ell} & P^{3}_{\alpha\gamma} \end{bmatrix} \tag{4-18}$$

$$B^{C12}_{p\beta} = \begin{bmatrix} B^{B12}_{m\beta} \\ P^{4}_{k\beta} \\ P^{5}_{\alpha\beta} \end{bmatrix} \tag{4-19}$$

$$B^{C13}_{p\beta} = \begin{bmatrix} B^{B13}_{m\beta} \\ B^{A13}_{k\beta} \\ P^{6}_{\alpha\beta} \end{bmatrix} \tag{4-20}$$

$$B^{C21}_{\alpha q} = \left[B^{B21}_{\alpha n} \;\middle|\; P^{7}_{\alpha \ell} \;\middle|\; P^{8}_{\alpha\gamma} \right] \tag{4-21}$$

$$B^{C22}_{\alpha\beta} = - P^{9}_{\alpha\gamma} \; P^{10}_{\gamma\beta} + P^{11}_{\alpha\gamma} \; A^{ji}_{\gamma\beta} + B^{B22}_{\alpha\beta} \tag{4-22}$$

$$B^{C23}_{\alpha\beta} = B^{B23}_{\alpha\beta} - P^{9}_{\alpha\beta} \tag{4-23}$$

$$B^{C31}_{\alpha q} = \left[B^{B31}_{\alpha n} \;\middle|\; A^{ji(T)}_{\alpha\beta} \; B^{A31}_{\beta\ell} \;\middle|\; P^{12}_{\alpha\beta} \; G^{j+}_{\beta\gamma} \right] \tag{4-24}$$

$$B^{C32}_{\alpha\beta} = B^{B32}_{\alpha\beta} + A^{ji(T)}_{\alpha\gamma} \; B^{A33}_{\gamma\delta} \; P^{10}_{\delta\beta} + P^{12}_{\alpha\gamma} \; A^{ji}_{\gamma\beta} \tag{4-25}$$

$$B^{C33}_{\alpha\beta} = B^{B33}_{\alpha\beta} + A^{ji(T)}_{\alpha\gamma} \; B^{A33}_{\gamma\beta} \tag{4-26}$$

$$C_p^{C1} = \begin{bmatrix} C_m^{B1} \\ P_k^{13} \\ P_\alpha^{14} \end{bmatrix} \tag{4-27}$$

$$C_\alpha^{C2} = C_\alpha^{B2} + P_{\alpha\beta}^{9} P_\beta^{16} - P_{\alpha\beta}^{11} P_\beta^{17} + A_{\alpha\beta}^{ji(T)} C_\beta^{A2} + P_{\alpha\beta}^{18} C_\beta^{A3} \tag{4-28}$$

$$C_\alpha^{C3} = C_\alpha^{B3} - P_{\alpha\beta}^{12} P_\beta^{17} + A_{\alpha\beta}^{ji(T)} \left\{ C_\beta^{A3} - B_{\beta\gamma}^{A33} P_\gamma^{16} \right\} \tag{4-29}$$

The $P_{k\beta}^{1}$ to $P_{\alpha\beta}^{18}$ are defined as follows:

$$P_{k\beta}^{1} = \left\{ B_{k\alpha}^{A12} + B_{k\gamma}^{A13} P_{\gamma\alpha}^{19} \right\} G_{\alpha\beta}^{j+} \tag{4-30}$$

where

$$P_{\beta\gamma}^{19} = A_{\beta\delta}^{jr(T)} \tilde{\ell}_{\delta\gamma}^{ji} \tag{4-31}$$

$$P_{\alpha\ell}^{2} = \left\{ B_{\alpha\ell}^{A21} - \tilde{\ell}_{\alpha\beta}^{ji} B_{\beta\ell}^{A31} \right\}^{\circ} \tag{4-32}$$

where the superscript 0 denotes removal of the αth row of the associated quantity if $\theta_\alpha^j \equiv 0$.

$$P_{\alpha\gamma}^{3} = \left\{ P_{\alpha\beta}^{20} G_{\beta\gamma}^{j+} \right\}^{\circ} \tag{4-33}$$

where the superscript 0 has the same meaning as above and

$$P_{\alpha\beta}^{20} = B_{\alpha\beta}^{A22} + B_{\alpha\gamma}^{A23} P_{\gamma\beta}^{19} - \tilde{\ell}_{\alpha\gamma}^{ji} \left\{ B_{\gamma\beta}^{A32} + B_{\gamma\delta}^{A33} P_{\delta\beta}^{19} \right\} . \tag{4-34}$$

$$P_{k\beta}^{4} = \left\{ B_{k\alpha}^{A12} + B_{k\gamma}^{A13} P_{\gamma\alpha}^{19} \right\} A_{\alpha\beta}^{ji} + B_{k\alpha}^{A13} P_{\alpha\beta}^{10} \tag{4-35}$$

$$P_{\alpha\beta}^{5} = \left\{ P_{\alpha\gamma}^{20} A_{\gamma\beta}^{ji} + P_{\alpha\gamma}^{21} P_{\gamma\beta}^{10} \right\}^{\circ} \tag{4-36}$$

where

$$P^{21}_{\alpha\beta} = B^{A23}_{\alpha\beta} - \tilde{\ell}^{ji}_{\alpha\gamma} \; B^{A33}_{\gamma\beta} \quad . \tag{4-37}$$

$$P^{6}_{\alpha\beta} = \left\{ P^{21}_{\alpha\beta} \right\}^{\circ} \tag{4-38}$$

$$P^{7}_{\alpha\ell} = P^{18}_{\alpha\beta} \; B^{A31}_{\beta\ell} + A^{ji(T)}_{\alpha\beta} \; B^{A21}_{\beta\ell} \tag{4-39}$$

$$P^{8}_{\alpha\beta} = P^{11}_{\alpha\gamma} \; G^{j+}_{\gamma\beta} \tag{4-40}$$

$$P^{9}_{\alpha\gamma} = - P^{18}_{\alpha\beta} \; B^{A33}_{\beta\gamma} - A^{ji(T)}_{\alpha\beta} \; B^{A23}_{\beta\gamma} \tag{4-41}$$

$$P^{10}_{\alpha\beta} = - A^{ir(T)}_{\alpha\gamma} \; \tilde{\ell}^{ij}_{\gamma\beta} \tag{4-42}$$

$$P^{11}_{\alpha\beta} = P^{18}_{\alpha\gamma} \; B^{A32}_{\gamma\beta} + A^{ji(T)}_{\alpha\gamma} \; B^{A22}_{\gamma\beta} - P^{9}_{\alpha\gamma} \; P^{19}_{\gamma\beta} \tag{4-43}$$

$$P^{12}_{\alpha\beta} = A^{ji(T)}_{\alpha\gamma} \left\{ B^{A32}_{\gamma\beta} + B^{A33}_{\gamma\delta} \; P^{19}_{\delta\beta} \right\} \tag{4-44}$$

$$P^{13}_{k} = C^{A1}_{k} - \left\{ B^{A12}_{k\alpha} + B^{A13}_{k\beta} \; P^{19}_{\beta\alpha} \right\} P^{17}_{\alpha} - B^{A13}_{k\beta} \; P^{16}_{\beta} \tag{4-45}$$

$$P^{14}_{\alpha} = \left\{ - P^{20}_{\alpha\beta} \; P^{17}_{\beta} - P^{21}_{\alpha\beta} \; P^{16}_{\beta} + C^{A2}_{\alpha} - \tilde{\ell}^{ji}_{\alpha\beta} \; C^{A3}_{\beta} + T^{ji}_{\alpha} \right\}^{\circ} \tag{4-46}$$

where the superscript $^{\circ}$ is again applied as before, meaning that only those components of T^{ji} along degree-of-freedom axes are pertinent

$$P^{16}_{\alpha} = A^{ir(T)}_{\alpha\beta} \left\{ \ddot{\ell}^{ij}_{\beta} + 2\,\tilde{\omega}^{i}_{\beta\gamma}\,\dot{\ell}^{ij}_{\gamma} + \tilde{\omega}^{i}_{\beta\gamma}\,\tilde{\omega}^{i}_{\gamma\delta}\,\ell^{ij}_{\delta} \right\} - A^{jr(T)}_{\alpha\beta} \left\{ \ddot{\ell}^{ji}_{\beta} + 2\,\tilde{\omega}^{j}_{\beta\gamma}\,\dot{\ell}^{ji}_{\gamma} + \tilde{\omega}^{j}_{\beta\gamma}\,\tilde{\omega}^{j}_{\gamma\delta}\,\ell^{ji}_{\delta} \right\} + \left\{ \dot{\tilde{\omega}}^{r}_{\alpha\beta} + \tilde{\omega}^{r}_{\alpha\gamma}\,\tilde{\omega}^{r}_{\gamma\beta} \right\} \left\{ R^{i}_{\beta} - R^{j}_{\beta} \right\} + 2\,\tilde{\omega}^{r}_{\alpha\beta} \left\{ \dot{R}^{i}_{\beta} - \dot{R}^{j}_{\beta} \right\} \tag{4-47}$$

$$P^{17}_{\alpha} = -\,\tilde{\omega}^{j}_{\alpha\beta}\,A^{ji}_{\beta\gamma}\,\omega^{i}_{\gamma} + \dot{G}^{j+}_{\alpha\beta}\,\dot{\theta}^{j^{\circ}}_{\beta} + P^{100}_{\alpha} \tag{4-48}$$

where

$$\dot{G}^{j}_{\alpha\beta} = \begin{bmatrix} (-\dot{\theta}^{j}_{2}\cos\theta^{j}_{3}\sin\theta^{j}_{2} - \dot{\theta}^{j}_{3}\sin\theta^{j}_{3}\cos\theta^{j}_{2}) & (\dot{\theta}^{j}_{3}\cos\theta^{j}_{3}) & 0 \\ (\dot{\theta}^{j}_{2}\sin\theta^{j}_{2}\sin\theta^{j}_{3} - \dot{\theta}^{j}_{3}\cos\theta^{j}_{3}\cos\theta^{j}_{2}) & (-\dot{\theta}^{j}_{3}\sin\theta^{j}_{3}) & 0 \\ (\dot{\theta}^{j}_{2}\cos\theta^{j}_{2}) & 0 & 0 \end{bmatrix} \tag{4-49}$$

the superscript $+$ again implying removal of the αth column if $\theta^{j}_{\alpha} \equiv 0$, and

$$P^{100}_{\alpha} = \left\{ A^{jjo}_{\alpha\beta}\,\dot{G}^{jo}_{\beta\sigma} + \left[A^{ji}_{\alpha\beta}\,\tilde{\omega}^{i}_{\beta\gamma}\,A^{joi(T)}_{\gamma\delta} - \tilde{\omega}^{j}_{\alpha\beta}\,A^{jjo}_{\beta\delta} \right] G^{jo}_{\delta\sigma} \right\} \dot{\theta}^{jo}_{\sigma} + A^{jjo}_{\alpha\beta}\,G^{jo}_{\beta\gamma}\,\ddot{\theta}^{jo}_{\gamma} \tag{4-50}$$

$$P^{18}_{\alpha\beta} = \tilde{\ell}^{ij}_{\alpha\gamma}\,A^{ji(T)}_{\gamma\beta} - A^{ji(T)}_{\alpha\gamma}\,\tilde{\ell}^{ji}_{\gamma\beta} \tag{4-51}$$

This completes the definition of the Rigid Combining Algorithm.

The kinematic equations for Body i (i≥2), that is, a determination at time t of its attitude rates, is already known since terms such as θ^{j}_{α} are already elements of the dynamic state vector. For Body 1, a direction cosine approach is desired and the kinematic equations are as follows:

$$\dot{A}^{lr}_{\alpha\beta} = - \tilde{\omega}^{l}_{\alpha\gamma} A^{lr}_{\gamma\beta} + A^{lr}_{\alpha\gamma} \tilde{\omega}^{r}_{\gamma\beta} \tag{4-52}$$

The above relations were coded in the initial UFSS Program. However, a much more efficient calculation of $A^{lr}_{\alpha\beta}$ is realized by utilizing an "Euler parameter" technique as follows.

Utilizing (4-52), it follows that

$$\dot{A}^{el}_{\alpha\beta} = A^{el}_{\alpha\gamma} \tilde{\omega}^{l}_{\gamma\beta} .$$

The initial orientation for $A^{el}_{\alpha\beta}$ is obtained from

$$A^{el}_{\alpha\beta} (t_o) = A^{er}_{\alpha\gamma}(t_o) A^{rl}_{\gamma\beta}(t_o) .$$

The matrix $A^{el}_{\alpha\beta}$ can be written in terms of $A^{el}_{\alpha\beta}(t_o)$, a constant, and a matrix $C_{\alpha\beta}$, a function of time as follows

$$A^{el}_{\alpha\beta}(t) = A^{el}_{\alpha\gamma}(t_o) C_{\gamma\beta}(t) , \tag{4-52-a}$$

so that $C_{\gamma\beta}(t)$ also satisfies the same differential equation as $A^{el}_{\alpha\beta}$; i.e.,

$$\dot{C}_{\alpha\beta} = C_{\alpha\gamma} \tilde{\omega}^{l}_{\gamma\beta} .$$

Instead of solving the above matrix differential equation, $C_{\alpha\beta}$ can be expressed in terms of the four Euler parameters consisting of a scaler χ and a vector κ_α resulting in four scalar differential equations to be integrated.

To introduce the Euler parameters, note that any orientation of a body may be achieved by a counterclockwise rotation about an appropriate unit vector e_α through an angle θ . Accordingly, $C_{\alpha\beta}$ has the representation

$$C_{\alpha\beta} = \delta_{\alpha\beta} + \sin\theta \, \tilde{e}_{\alpha\beta} + (1 - \cos\theta) \, \tilde{e}_{\alpha\gamma} \, \tilde{e}_{\gamma\beta} .$$

The Euler parameters are defined in terms of e_α and θ by

$$\chi = \cos(\theta/2)$$

$$\kappa_\alpha = \sin(\theta/2)\, e_\alpha \, .$$

Then

$$C_{\alpha\beta} = \delta_{\alpha\beta} + 2\,\chi\,\tilde{\kappa}_{\alpha\beta} + 2\,\tilde{\kappa}_{\alpha\gamma}\,\tilde{\kappa}_{\gamma\beta} \tag{4-52-b}$$

The differential equation for $C_{\alpha\beta}$ leads to corresponding differential equations for χ and κ_α, namely

$$\begin{aligned} \dot{\chi} &= -1/2\ \omega_\alpha^{1(T)}\ \kappa_\alpha \\ \dot{\kappa}_\alpha &= 1/2\ \chi\left(\omega_\alpha^1 - \tilde{\omega}_{\alpha\beta}^1\,\kappa_\beta\right) \, . \end{aligned} \tag{4-52-c}$$

By definition, $\chi^2 + \kappa_\alpha^{(T)}\ \kappa_\alpha$ is equal to one and indeed this function is an integral of (4-52-c). This fact is used to provide a check on the computation through calculation of

$$\Delta = \left|1 - \chi^2 + \kappa_\alpha^{(T)}\ \kappa_\alpha\right| \, . \tag{4-52-d}$$

As seen from (4-52-a), the initial value of $C_{\alpha\beta}$ is the identity matrix. Thus, from (4-52-b), initial conditions for (4-52-c) are

$$\chi(t_o) = 1 \quad ; \quad \kappa_\alpha(t_o) = 0 \, . \tag{4-52-e}$$

Finally, the equations (4-52-c) and (4-52-e) are solved for $\chi(t)$ and $\kappa_\alpha(t)$ with (4-52-b) subsequently used to obtain $C_{\alpha\beta}(t)$. From (4-52-a)

$$A_{\alpha\beta}^{1e}(t) = C_{\alpha\gamma}^{(T)}(t)\ A_{\gamma\beta}^{1e}(t_o) \, .$$

Thus, since $A_{\alpha\beta}^{1e} = A_{\alpha\gamma}^{1r}\ A_{\gamma\beta}^{re}$, $A_{\alpha\beta}^{1r}$ is obtained from

$$A^{1r}_{\alpha\beta}(t) = C^{(T)}_{\alpha\gamma}(t)\, A^{1r}_{\gamma\sigma}(t_o)\, A^{re}_{\sigma\rho}(t_o)\, A^{re(T)}_{\rho\beta}(t). \tag{4-52-f}$$

It now remains to detail the initialization of the inductive process.

4.3 Initialization of the Rigid Combining Algorithm

When one of the inputs to the Rigid Combining Algorithm is a single rigid body (say Body k is the System A), then the matrices B^A and C^A are initialized as follows:

$$B^{A22}_{\alpha\beta} = I^k_{\alpha\beta} \tag{4-53}$$

$$B^{A33}_{\alpha\beta} = m^k \; A^{kr}_{\alpha\beta} \tag{4-54}$$

$$B^{A23}_{\alpha\beta} = B^{A32}_{\alpha\beta} = 0 \tag{4-54.1}$$

All the remaining sub-matrices of B^A are void. In addition,

$$C^{A1}_{\alpha} \text{ is void} \tag{4-55}$$

$$C^{A2}_{\alpha} = - \tilde{\omega}^k_{\alpha\gamma}\, I^k_{\gamma\delta}\, \omega^k_{\delta} + T^{ke}_{\alpha} \tag{4-56}$$

$$C^{A3}_{\alpha} = - m^k \; A^{kr}_{\alpha\gamma} \left[2\, \tilde{\omega}^r_{\gamma\delta}\, \dot{R}^k_{\delta} + (\dot{\tilde{\omega}}^r_{\gamma\delta} + \tilde{\omega}^r_{\gamma\beta}\, \tilde{\omega}^r_{\beta\delta})\, R^k_{\delta} \right] + F^{ke}_{\alpha} \tag{4-57}$$

If Body k is, instead, a single rigid body treated as a limb (System B), then the above specified sub-matrices of B^A and C^A define the sub-matrices of B^B and C^B. In either event, the Rigid Combining Algorithm can be readily initialized in its inductive procedure of synthesizing the system dynamic equations. For specification of T^{ke}_{α} and F^{ke}_{α}, see Section 6.1.

4.4 Specification of the Torque T^{ji}_{α} (needed in (4-46))

The relative rotations θ^j_α of Body j with respect to its limb, Body i, are assumed to take place about a gimballed hinge nominally aligned with the axis frame $(x^{jo}_1, x^{jo}_2, x^{jo}_3)$. The components of the torque T^{ji}_α are functions of the reaction torques at these gimbal axes.

Specifically, let T^{jh}_1 , T^{jh}_2 , T^{jh}_3 be the gimbal hinge reaction torques about the x^g_1 , x^g_2 , x^g_3 gimbal axes of Body j. These three gimbal torques may be arranged in a 3 x 1 column matrix such that

$$T^{jh}_\alpha = \begin{Bmatrix} T^{jh}_1 \\ T^{jh}_2 \\ T^{jh}_3 \end{Bmatrix} \tag{4-58}$$

where the elements of T^{jh}_α are not orthogonal components of a resultant torque vector.

It can readily be shown that the torque components of the total gimbal reaction torque, transmitted from Body i to Body j, defined in Body j coordinates are given by

$$T^{ji}_\alpha = \left[G^j(T)\right]^{-1}_{\alpha\beta} T^{jh}_\beta \tag{4-59}$$

where

$$\left[G^j(T)\right]^{-1}_{\alpha\beta} = \begin{bmatrix} \dfrac{\cos\theta^j_3}{\cos\theta^j_2} & \sin\theta^j_3 & -\dfrac{\sin\theta^j_2 \cos\theta^j_3}{\cos\theta^j_2} \\ -\dfrac{\sin\theta^j_3}{\cos\theta^j_2} & \cos\theta^j_3 & \dfrac{\sin\theta^j_2 \sin\theta^j_3}{\cos\theta^j_2} \\ 0 & 0 & 1 \end{bmatrix} \tag{4-60}$$

(It is assumed that the gimbal rotations are such that $0 \leq \theta_2^j < \pi/2$ so that $\cos \theta_2^j \neq 0$.)

The gimbal torques about the gimbal axes of Body j are

$$T_\alpha^{jh} = T_\alpha^{js} + T_\alpha^{jd} + T_\alpha^{jm} \tag{4-61}$$

where

$$T_\alpha^{js} = - K_{\alpha\beta}^{j1} \theta_\beta^j - K_{\alpha\beta}^{j2} \theta_\beta^{j2} - K_{\alpha\beta}^{j3} \theta_\beta^{j3}$$

are the spring restraint torques,

$$T_\alpha^{jd} = - C_{\alpha\beta}^{j1} \dot{\theta}_\beta^j - C_{\alpha\beta}^{j2} \dot{\theta}_\beta^{j2} - C_{\alpha\beta}^{j3} \dot{\theta}_\beta^{j3}$$

are the damping torques and

T_α^{jm} is the array containing the motor torques about the gimbal axes of Body j in case Body j is a controlled body.

In the above relations,

$$K_{\alpha\beta}^{jn} = \begin{bmatrix} k_1^{jn} & 0 & 0 \\ 0 & k_2^{jn} & 0 \\ 0 & 0 & k_3^{jn} \end{bmatrix}$$

where k_1^{jn}, k_2^{jn}, k_3^{jn} are the spring constants associated with the x_1^g, x_2^g, x_3^g gimbal axes of Body j in units of $FL/(rad)^n$;

$$C_{\alpha\beta}^{jn} = \begin{bmatrix} c_1^{jn} & 0 & 0 \\ 0 & c_2^{jn} & 0 \\ 0 & 0 & c_3^{jn} \end{bmatrix}$$

where c_1^{jn}, c_2^{jn}, c_3^{jn} are the damping constants associated with the x_1^g, x_2^g, x_3^g gimbal axes of Body j in units of $FL/(rad/T)^n$. In addition, the following notational convenience is utilized:

$$\theta_\alpha^{jn} = \begin{Bmatrix} (\theta_1^j)^n \\ (\theta_2^j)^n \\ (\theta_3^j)^n \end{Bmatrix} .$$

The body j control motor torques T_α^{jm} are computed in the control routines.

V. Flexible Combining Algorithm Specification

Consider now the case where System A in a given combining operation is a single flexible body. In this case, the Flexible Combining Algorithm is utilized to synthesize the dynamic equations for System C given those for Systems A and B as shown in Figure 5.1. Prior to the first pass through this algorithm, the following quantities are computed.

5.1 Auxiliary Flexible Quantities

The following quantities are computed for each flexible body at the time the auxiliary variables are calculated:

$$H_{k\alpha}^{j1} = q_\ell^j \; Z_{k\ell\alpha}^j \tag{5-1}$$

$$H_{k\alpha}^{j2} = \dot{q}_\ell^j \; Z_{k\ell\alpha}^j \tag{5-2}$$

$$H_\alpha^{j3} = q_k^j \; \Phi_{k\alpha}^j \tag{5-3}$$

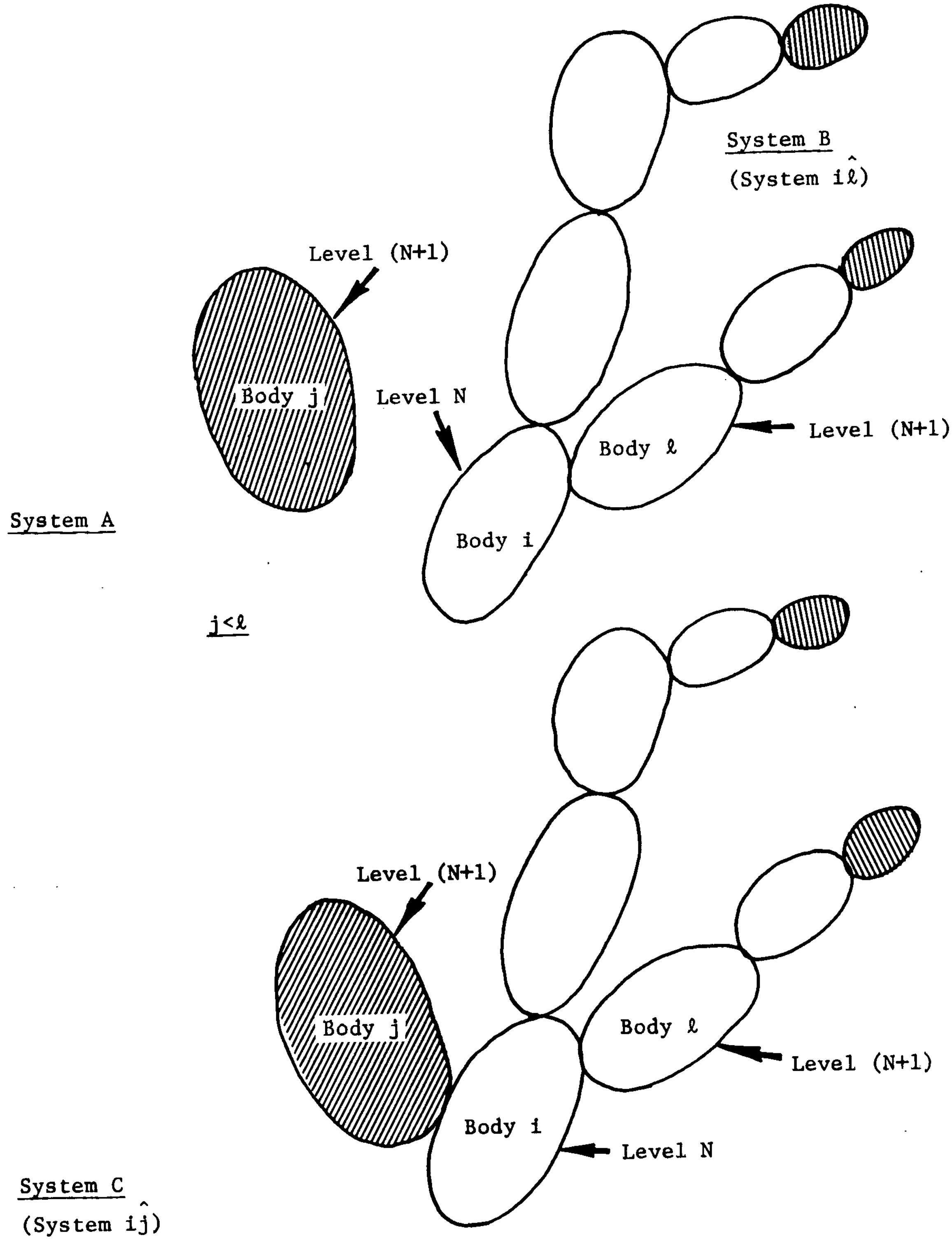


Figure 5.1. Use of the Flexible Combining Algorithm

$$H^{j4}_{\alpha} = \dot{q}^j_k \; \Phi^j_{k\alpha} \tag{5-4}$$

$$H^{j5}_{k\alpha\beta} = q^j_\ell \; E^j_{k\ell\alpha\beta} \tag{5-5}$$

$$H^{j6}_{k\alpha} = q^j_\ell \; Z^j_{\ell k\alpha} = - H^{j1}_{k\alpha} \tag{5-6}$$

$$H^{j7}_{\alpha} = \dot{q}^j_k \; \dot{q}^j_\ell \; Z^j_{k\ell\alpha} \tag{5-7}$$

$$H^{j8}_{\alpha\beta} = H^{j11}_{\alpha\beta} + H^{j11(T)}_{\alpha\beta} \tag{5-8}$$

$$H^{j9}_{\alpha\beta} = \dot{q}^j_k \; q^j_\ell \; E^j_{k\ell\alpha\beta} = \dot{q}^j_k \; H^{j5}_{k\alpha\beta} \tag{5-9}$$

$$H^{j10}_{\alpha\beta} = q^j_k \; q^j_\ell \; E^j_{k\ell\alpha\beta} = q^j_k \; H^{j5}_{k\alpha\beta} \tag{5-10}$$

$$H^{j11}_{\alpha\beta} = q^j_\ell \; N^j_{\ell\alpha\beta} \tag{5-11}$$

$$H^{j12}_{\alpha\beta} = \dot{q}^j_\ell \; N^j_{\ell\alpha\beta} \tag{5-12}$$

In all the above quantities,

$$\alpha,\beta = 1,2,3$$

$$k,\ell = 1,2,\ldots,n_j \quad .$$

The arrays $Z^j_{k\ell\alpha}$, $\Phi^j_{k\alpha}$, $E^j_{k\ell\alpha\beta}$ and $N^j_{\ell\alpha\beta}$ are obtained from the Mass Properties Subroutine and will be defined in Section IX.

5.2 The Flexible Combining Algorithm

The Flexible Combining Algorithm is used only when System A in a given combining operation is a single flexible body, call it Body j. Once again, assume that System B has Body i as its member of lowest level ($i<j$) and that System A is to be connected to System B to yield System C as in Figure 5.1. Referring to this figure, it is also assumed that Body ℓ is the lowest numbered branch of Body i in System B and that $j<\ell$.

The dynamic equations for System B will be specified first since they are identical to the System B equations for the Rigid Combining Algorithm.

<u>System B</u>

$$\begin{bmatrix} B^{B11}_{mn} & B^{B12}_{m\beta} & B^{B13}_{m\beta} \\ B^{B21}_{\alpha n} & B^{B22}_{\alpha\beta} & B^{B23}_{\alpha\beta} \\ B^{B31}_{\alpha n} & B^{B32}_{\alpha\beta} & B^{B33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{B1}_{m} \\ C^{B2}_{\alpha} \\ C^{B3}_{\alpha} \end{bmatrix} \tag{5-13}$$

Once again, $B^{B22}_{\alpha\beta}$, $B^{B23}_{\alpha\beta}$, $B^{B32}_{\alpha\beta}$ and $B^{B33}_{\alpha\beta}$ are 3 x 3 matrices with the indices α and β running from 1 to 3; C^{B2}_{α} and C^{B3}_{α} are 3 x 1 matrices; if $\ddot{\theta}^{i\hat{\ell}}_{n}$ has r components ($n=1,2,\ldots,r$), then B^{B11}_{mn} is r x r, $B^{B12}_{m\beta}$ and $B^{B13}_{m\beta}$ are r x 3, $B^{B21}_{\alpha n}$ and $B^{B31}_{\alpha n}$ are 3 x r while C^{B1}_{m} is r x 1.

Specification of the System A equations will now be given. It should be noted that since System A is a single body, the following form of System A occurs in the synthesizing algorithm only as a matrix loading operation and never as the result of a combining operation.

<u>System A</u>

$$\begin{bmatrix} B^{A11}_{k\ell} & B^{A12}_{k\gamma} & B^{A13}_{k\beta} & B^{A14}_{k\beta} \\ B^{A21^\circ}_{\delta\ell} & B^{A22^\circ}_{\delta\gamma} & B^{A23^\circ}_{\delta\beta} & B^{A24^\circ}_{\delta\beta} \\ B^{A31}_{\alpha\ell} & B^{A32}_{\alpha\gamma} & B^{A33}_{\alpha\beta} & B^{A34}_{\alpha\beta} \end{bmatrix} \cdot \begin{Bmatrix} \ddot{q}^{j}_{\ell} \\ \ddot{\theta}^{j}_{\gamma} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{Bmatrix} = \begin{Bmatrix} C^{A1}_{k} \\ C^{A2^\circ}_{\delta} \\ C^{A3}_{\alpha} \end{Bmatrix} \tag{5-14}$$

Here,

$$k,\ell = 1,2,\ldots,n_j$$

$$\delta,\gamma = 0,1,\ldots,p_j$$

$$\alpha,\beta = 1,2,3$$

Thus, $B^{A11}_{k\ell}$ is an $n_j \times n_j$ matrix; $B^{A12}_{k\gamma}$ is $n_j \times p_j$; $B^{A13}_{k\beta}$ and $B^{A14}_{k\beta}$ are $n_j \times 3$; C^{A1}_{k} is $n_j \times 1$; $B^{A21°}_{\delta\ell}$ is $p_j \times n_j$; $B^{A22°}_{\delta\gamma}$ is $p_j \times p_j$; $B^{A23°}_{\delta\beta}$ and $B^{A24°}_{\delta\beta}$ are $p_j \times 3$; $C^{A2°}_{\delta}$ is $p_j \times 1$; $B^{A31}_{\alpha\ell}$ is $3 \times n_j$; $B^{A32}_{\alpha\gamma}$ is $3 \times p_j$; $B^{A33}_{\alpha\beta}$ and $B^{A34}_{\alpha\beta}$ are 3×3 ; C^{A3}_{α} is 3×1.

(Note here that the sub-matrices occurring in the equation for θ^j_γ appear with a superscript ° denoting removal of the λth row of the sub-matrices if $\theta^j_\lambda \equiv 0$; hence, δ and γ need not run fully from 1 to 3 as must α and β . However, the full sub-matrices must be loaded as they appear in the System C equations except for Equations 5.17-5.20)

Finally, the output of the Flexible Combining Algorithm has a form identical to the output of the Rigid Combining Algorithm.

<u>System C</u>

$$\begin{bmatrix} B^{C11}_{ab} & B^{C12}_{a\beta} & B^{C13}_{a\beta} \\ B^{C21}_{\alpha b} & B^{C22}_{\alpha\beta} & B^{C23}_{\alpha\beta} \\ B^{C31}_{\alpha b} & B^{C32}_{\alpha\beta} & B^{C33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{j}}_{b} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{C1}_{a} \\ C^{C2}_{\alpha} \\ C^{C3}_{\alpha} \end{bmatrix} \tag{5-15}$$

where

$$\alpha, \beta = 1,2,3$$

$$a, b = 1,2,\ldots,(r + n_j + p_j)$$

and

$$\ddot{\theta}^{i\hat{j}}_{b} = \begin{Bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \ddot{q}^{j}_{\ell} \\ \ddot{\theta}^{j}_{\gamma} \end{Bmatrix} \tag{5-16}$$

again with the column matrix $\ddot{\theta}^j_\gamma$ including only the degree-of-freedom components at the interconnection of Bodies i and j.

To complete the description of the Flexible Combining Algorithm, it is now necessary to explicitly specify the elements of the matrices B^C and C^C and to show how the elements of the matrices B^A and C^A are initialized. (Note that if System B is a single rigid body, it is initialized as shown in Section 4.3.)

$$B^{C11}_{ab} = \begin{bmatrix} B^{B11}_{mn} & 0 & 0 \\ 0 & B^{A11}_{k\ell} & B^{A12}_{k\gamma} \\ 0 & B^{A21°}_{\delta\ell} & B^{A22°}_{\delta\gamma} \end{bmatrix} \tag{5-17}$$

$$B^{C12}_{a\beta} = \begin{bmatrix} B^{B12}_{m\beta} \\ B^{A13}_{k\beta} \\ B^{A23°}_{\delta\beta} \end{bmatrix} \tag{5-18}$$

$$B^{C13}_{a\beta} = \begin{bmatrix} B^{B13}_{m\beta} \\ B^{A14}_{k\beta} \\ B^{A24°}_{\delta\beta} \end{bmatrix} \tag{5-19}$$

$$C^{C1}_{a} = \begin{bmatrix} C^{B1}_{m} \\ C^{A1}_{k} \\ C^{A2°}_{\delta} + T^{ji°}_{\delta} \end{bmatrix} \tag{5-20}$$

$$B^{C21}_{\alpha b} = \left[B^{B21}_{\alpha n} \;\middle|\; Q^{1}_{\alpha \ell} \;\middle|\; Q^{2}_{\alpha \gamma} \right] \tag{5-21}$$

$$B^{C22}_{\alpha\beta} = Q^{3}_{\alpha\beta} \tag{5-22}$$

$$B^{C23}_{\alpha\beta} = Q^{4}_{\alpha\beta} \tag{5-23}$$

$$C^{C2}_{\alpha} = Q^{5}_{\alpha} \tag{5-24}$$

$$B^{C31}_{\alpha b} = \left[B^{B31}_{\alpha n} \;\middle|\; A^{ir}_{\alpha\sigma} \; B^{A31}_{\sigma\ell} \;\middle|\; A^{ir}_{\alpha\sigma} \; B^{A32}_{\sigma\gamma} \right] \tag{5-25}$$

$$B^{C32}_{\alpha\beta} = B^{B32}_{\alpha\beta} + A^{ir}_{\alpha\sigma} \; B^{A33}_{\sigma\beta} \tag{5-26}$$

$$B^{C33}_{\alpha\beta} = B^{B33}_{\alpha\beta} + A^{ir}_{\alpha\sigma} \; B^{A34}_{\sigma\beta} \tag{5-27}$$

$$C^{C3}_{\alpha} = C^{B3}_{\alpha} + A^{ir}_{\alpha\beta} \; C^{A3}_{\beta} \quad . \tag{5-28}$$

The Q^1 to Q^5 are defined as follows:

$$Q^{1}_{\alpha\ell} = A^{ji(T)}_{\alpha\sigma} \; B^{A21}_{\sigma\ell} + \tilde{\ell}^{ij}_{\alpha\sigma} \; A^{ir}_{\sigma\rho} \; B^{A31}_{\rho\ell} \tag{5-29}$$

$$Q^{2}_{\alpha\gamma} = A^{ji(T)}_{\alpha\sigma} \; B^{A22}_{\sigma\gamma} + \tilde{\ell}^{ij}_{\alpha\sigma} \; A^{ir}_{\sigma\rho} \; B^{A32}_{\rho\gamma} \tag{5-30}$$

$$Q^{3}_{\alpha\beta} = B^{B22}_{\alpha\beta} + A^{ji(T)}_{\alpha\sigma} \; B^{A23}_{\sigma\beta} + \tilde{\ell}^{ij}_{\alpha\sigma} \; A^{ir}_{\sigma\rho} \; B^{A33}_{\rho\beta} \tag{5-31}$$

$$Q^{4}_{\alpha\beta} = B^{B23}_{\alpha\beta} + A^{ji(T)}_{\alpha\sigma} \; B^{A24}_{\sigma\beta} + \tilde{\ell}^{ij}_{\alpha\sigma} \; A^{ir}_{\sigma\rho} \; B^{A34}_{\rho\beta} \tag{5-32}$$

$$Q^{5}_{\alpha} = C^{B2}_{\alpha} + A^{ji(T)}_{\alpha\beta} \; C^{A2}_{\beta} + \tilde{\ell}^{ij}_{\alpha\sigma} \; A^{ir}_{\sigma\beta} \; C^{A3}_{\beta} \tag{5-33}$$

This completes the definition of the Flexible Combining Algorithm. It now remains only to detail the initialization of Systems A and B for this combining algorithm. However, before so doing, the expanded form of System C will be presented for completeness.

$$
\left[\begin{array}{ccc|c|c}
B^{B11}_{mn} & 0 & 0 & B^{B12}_{m\beta} & B^{B13}_{m\beta} \\
0 & B^{A11}_{k\ell} & B^{A12}_{k\gamma} & B^{A13}_{k\beta} & B^{A14}_{k\beta} \\
0 & B^{A21\circ}_{\delta\ell} & B^{A22\circ}_{\delta\gamma} & B^{A23\circ}_{\delta\beta} & B^{A24\circ}_{\delta\beta} \\
\hdashline
B^{B21}_{\alpha n} & Q^{1}_{\alpha\ell} & Q^{2}_{\alpha\gamma} & Q^{3}_{\alpha\beta} & Q^{4}_{\alpha\beta} \\
\hdashline
\left(B^{B31}_{\alpha n}\right) & \left(A^{ir}_{\alpha\sigma}\ B^{A31}_{\sigma\ell}\right) & \left(A^{ir}_{\alpha\sigma}\ B^{A32}_{\sigma\gamma}\right) & B^{B32}_{\alpha\beta} + A^{ir}_{\alpha\sigma}\ B^{A33}_{\sigma\beta} & B^{B33}_{\alpha\beta} + A^{ir}_{\alpha\sigma}\ B^{A34}_{\sigma\beta}
\end{array}\right]
\left[\begin{array}{c}
\ddot{\theta}^{i\hat{\ell}}_{n} \\
\ddot{q}^{j}_{\ell} \\
\ddot{\theta}^{j}_{\gamma} \\
\hdashline
\dot{\omega}^{i}_{\beta} \\
\hdashline
\ddot{R}^{i}_{\beta}
\end{array}\right]
=
$$

$$
= \left[\begin{array}{c}
C^{B1}_{m} \\
C^{A1}_{k} \\
C^{A2\circ}_{\delta} + T^{ji\circ}_{\delta} \\
\hdashline
Q^{5}_{\alpha} \\
\hdashline
C^{B3}_{\alpha} + A^{ir}_{\alpha\beta}\ C^{A3}_{\beta}
\end{array}\right] \tag{5-34}
$$

5.3 Initialization of the Flexible Combining Algorithm

As stated previously, if System B is a single rigid body, then its initialization is identical to that of the Rigid Combining Algorithm presented in Section 4.3. Initialization or loading of System A is as follows:

$$B^{A11}_{k\ell} = m^{j}\ M^{j}_{k\ell} \tag{5-35}$$

$$B^{A12}_{k\gamma} = m^{j}\ S^{j4}_{k\sigma}\ G^{j+}_{\sigma\gamma} \tag{5-36}$$

$$B^{A13}_{k\beta} = m^{j}\ S^{j4}_{k\sigma}\ A^{ji}_{\sigma\beta} - m^{j}\ \Phi^{j}_{k\sigma}\ A^{ji}_{\sigma\rho}\ \tilde{\ell}^{ij}_{\rho\beta} \tag{5-37}$$

$$B^{A14}_{k\beta} = m^j \; \Phi^j_{k\sigma} \; A^{jr}_{\sigma\beta} \tag{5-38}$$

$$\begin{aligned} C^{A1}_k = & - m^j \Big\{ S^{j4}_{k\beta} \; S^{j5}_\beta + \Phi^j_{k\beta} \; S^{j7}_\beta \\ & - 2 \, H^{j2}_{k\beta} \; \omega^j_\beta - S^{j10}_k \\ & + K^j_{k\ell} \; q^j_\ell + V^j_{k\ell} \; \dot{q}^j_\ell \Big\} + Q^{je}_k \end{aligned} \tag{5-39}$$

where $M^j_{k\ell}$ and $K^j_{k\ell}$ are respectively the Body j generalized mass and stiffness matrices obtained from the Mass Properties Subroutine while $V^j_{k\ell}$ is the input generalized modal damping matrix.

$$B^{A21}_{\alpha\ell} = m^j \; S^{j8}_{\alpha\ell} \tag{5-40}$$

$$B^{A22}_{\alpha\gamma} = m^j \; S^{j9}_{\alpha\sigma} \; G^{j+}_{\sigma\gamma} \tag{5-41}$$

$$B^{A23}_{\alpha\beta} = m^j \; S^{j9}_{\alpha\sigma} \; A^{ji}_{\sigma\beta} - m^j \; \tilde{S}^{j6}_{\alpha\sigma} \; A^{ji}_{\sigma\rho} \; \tilde{\ell}^{ij}_{\rho\beta} \tag{5-42}$$

$$B^{A24}_{\alpha\beta} = m^j \; \tilde{S}^{j6}_{\alpha\sigma} \; A^{jr}_{\sigma\beta} \tag{5-43}$$

$$\begin{aligned} C^{A2}_\alpha = & - m^j \Big\{ S^{j9}_{\alpha\beta} \; S^{j5}_\beta + \tilde{S}^{j6}_{\alpha\beta} \; S^{j7}_\beta + \tilde{\omega}^j_{\alpha\sigma} \; S^{j9}_{\sigma\beta} \; \omega^j_\beta \\ & + 2 \; S^{j15}_{\alpha\beta} \; \omega^j_\beta \Big\} + H^{je}_\alpha \end{aligned} \tag{5-44}$$

$$B^{A31}_{\alpha\ell} = m^j \; A^{jr(T)}_{\alpha\sigma} \; \Phi^{j(T)}_{\sigma\ell} \tag{5-45}$$

$$B^{A32}_{\alpha\gamma} = - m^j \; A^{jr(T)}_{\alpha\sigma} \; S^{j6}_{\sigma\rho} \; G^{j+}_{\rho\gamma} \tag{5-46}$$

$$B^{A33}_{\alpha\beta} = - m^j \; A^{jr(T)}_{\alpha\sigma} \; \tilde{S}^{j6}_{\sigma\rho} \; A^{ji}_{\rho\beta} - m^j \; A^{ir(T)}_{\alpha\sigma} \; \tilde{\ell}^{ij}_{\sigma\beta} \tag{5-47}$$

$$B^{A34}_{\alpha\beta} = m^j \; \delta_{\alpha\beta} \tag{5-48}$$

$$C_{\alpha}^{A3} = - m^{j} \; A_{\alpha\sigma}^{jr(T)} \left\{ S_{\sigma}^{j7} - \tilde{S}_{\sigma\beta}^{j6} \; S_{\beta}^{j5} \right.$$

$$\left. + \tilde{\omega}_{\sigma\rho}^{j} \; \tilde{\omega}_{\rho\beta}^{j} \; S_{\beta}^{j6} + 2 \; \tilde{\omega}_{\rho\beta}^{j} \; H_{\beta}^{j4} \right\} \tag{5-49}$$

$$+ R_{\alpha}^{je}$$

In the above equations, Q_k^{je}, $H_{\dot{\alpha}}^{je}$ and R_{α}^{je} are external generalized forces to be specified in Section VII. Finally, the S^{j1} through S^{j10} are defined as follows (these quantities are computed for all flexible bodies at the time the auxiliary variables are calculated).

$$S_{\alpha}^{j1} = A_{\alpha\sigma}^{jr} \left(2 \, \tilde{\omega}_{\sigma\beta}^{r} \; \dot{R}_{\beta}^{i} + \dot{\tilde{\omega}}_{\sigma\beta}^{r} \; R_{\beta}^{i} + \tilde{\omega}_{\sigma\rho}^{r} \; \tilde{\omega}_{\rho\beta}^{r} \; R_{\beta}^{i} \right)$$

$$+ A_{\alpha\sigma}^{ji} \; \tilde{\omega}_{\sigma\rho}^{i} \; \tilde{\omega}_{\rho\beta}^{i} \; \ell_{\beta}^{ij} \tag{5-50}$$

$$S_{\alpha}^{j2} = \ddot{\ell}_{\alpha}^{ij} + 2 \; \tilde{\omega}_{\alpha\beta}^{i} \; \dot{\ell}_{\beta}^{ij} \tag{5-51}$$

$$S_{\alpha}^{j3} = \left[A_{\alpha\sigma}^{jjo} \; \dot{G}_{\sigma\beta}^{jo} + \left(A_{\alpha\sigma}^{ji} \; \tilde{\omega}_{\sigma\rho}^{i} \; A_{\rho\varepsilon}^{joi(T)} - \tilde{\omega}_{\alpha\sigma}^{j} \; A_{\sigma\varepsilon}^{jjo} \right) G_{\varepsilon\beta}^{jo} \right] \dot{\theta}_{\beta}^{jo}$$

$$+ A_{\alpha\sigma}^{jjo} \; G_{\sigma\beta}^{jo} \; \ddot{\theta}_{\beta}^{jo} = P_{\alpha}^{100} \quad \text{[see (4-50)]} \tag{5-52}$$

$$S_{k\beta}^{j4} = Y_{k\beta}^{j} - H_{k\beta}^{j1} \tag{5-53}$$

$$S_{\alpha}^{j5} = - \tilde{\omega}_{\alpha\sigma}^{j} \; A_{\sigma\beta}^{ji} \; \omega_{\beta}^{i} + \dot{G}_{\alpha\gamma}^{j+} \; \dot{\theta}_{\gamma}^{jo} + S_{\alpha}^{j3} \equiv P_{\alpha}^{17} \tag{5-54}$$

$$S_{\alpha}^{j6} = d_{\alpha}^{j} + H_{\alpha}^{j3} \tag{5-55}$$

$$S_{\alpha}^{j7} = S_{\alpha}^{j1} + A_{\alpha\beta}^{ji} \; S_{\beta}^{j2} \tag{5-56}$$

$$S_{\alpha\ell}^{j8} = Y_{\alpha\ell}^{j(T)} + H_{\alpha\ell}^{j6(T)} \equiv S_{\alpha\ell}^{j4(T)} \tag{5-57}$$

$$S_{\alpha\beta}^{j9} = I_{\alpha\beta}^{jf} + H_{\alpha\beta}^{j8} + H_{\alpha\beta}^{j10} \tag{5-58}$$

$$S_{k}^{j10} = (N_{k\alpha\beta}^{j} + H_{k\alpha\beta}^{j5}) \; \omega_{\alpha}^{j} \; \omega_{\beta}^{j} \tag{5-59}$$

In addition, the following quantities will be needed for the disturbance calculations to follow and should be calculated with the above:

$$S^{j11}_{k\alpha} = a^{r(T)}_{\beta} \; A^{je(T)}_{\beta\delta} \; B^{j}_{k\alpha\delta} \tag{5-60}$$

$$S^{j12}_{k\ell\beta} = a^{r(T)}_{\delta} \; A^{je(T)}_{\delta\alpha} \; C^{j}_{k\ell\alpha\beta} \tag{5-61}$$

$$S^{j13}_{\ell\alpha} = q^{j}_{k} \; S^{j12}_{k\ell\alpha} \tag{5-62}$$

$$S^{j14}_{k\alpha} = B^{j}_{k\alpha\beta} \; A^{je}_{\beta\rho} \; a^{r}_{\rho} \tag{5-63}$$

$$S^{j15}_{\alpha\beta} = H^{j12(T)}_{\alpha\beta} + H^{j9}_{\alpha\beta} \tag{5-64}$$

Once again the arrays $B^{j}_{k\beta\alpha}$ and $C^{j}_{k\ell\alpha\beta}$ are obtained from the Mass Properties Subroutine, while a^{r}_{ρ} is determined by the Orbit Subroutine.

VI. Distubance Subroutine

The Disturbance and Control Subroutines supply the perturbing forces and torques required in Equations (4-56), (4-57), (5-39), (5-44) and (5-49). At present, the Control Subroutine is not implemented, although the Generalized Control Interface Routine described in Section VIII is an integral part of the program.

The general disturbance equations are as follows:

$$T_{\alpha}^{je} = T_{\alpha}^{jD} + T_{\alpha}^{jC} \tag{6-1}$$

$$F_{\alpha}^{je} = F_{\alpha}^{jD} + F_{\alpha}^{jC} \tag{6-2}$$

$$Q_{k}^{je} = Q_{k}^{jD} + Q_{k}^{jC} \tag{6-3}$$

$$R_{\alpha}^{je} = R_{\alpha}^{jD} + R_{\alpha}^{jC} \tag{6-4}$$

$$H_{\alpha}^{je} = H_{\alpha}^{jD} + H_{\alpha}^{jC} \tag{6-5}$$

where all the superscript ^{jC} quantities are presently void control vector forces and torques and

$$T_{\alpha}^{jD} = T_{\alpha}^{jA} + T_{\alpha}^{jS} + T_{\alpha}^{jM} + T_{\alpha}^{jG} + T_{\alpha}^{jP} \tag{6-6}$$

$$F_{\alpha}^{jD} = F_{\alpha}^{jA} + F_{\alpha}^{jS} + F_{\alpha}^{jS} + F_{\alpha}^{jG} + F_{\alpha}^{jP} \tag{6-7}$$

$$Q_{k}^{jD} = Q_{k}^{jA} + Q_{k}^{jS} + Q_{k}^{jG} \tag{6-8}$$

$$R_{\alpha}^{jD} = R_{\alpha}^{jA} + R_{\alpha}^{jS} + R_{\alpha}^{jG} \tag{6-9}$$

$$H_{\alpha}^{jD} = H_{\alpha}^{jA} + H_{\alpha}^{jS} + H_{\alpha}^{jG} \tag{6-10}$$

Definitions of the individual terms in the above relations are amply provided by the flow-diagram of Figure 6.1 as well as by the table of Section X.

The following sub-sections present the equations necessary for programming the disturbances listed above (other than control disturbances). Section 6.1 details initial expressions for the prescribed disturbances. Section 6.2 (pages 45 through 60) details the environmental disturbance equations for rigid bodies, while Section 6.3 presents the environmental disturbance equations for flexible bodies. Since the rigid-body environmental disturbances are identical to those found in the TRW GSS program, their derivation can be found in Reference 1 and is not included in this document. However, the derivation of the flexible-body environmental disturbances is presented in Appendix B.

6.1 Prescribed Disturbances

Prescribed forces and torques are allowed to act about the Body j mass center. Specifically, the following analytical representation is utilized initially: (t_r is a reference time, normally equal to zero)

$$F_\alpha^{jP} = C_\alpha^{jf1} + C_\alpha^{jf2}(t - t_r) + C_\alpha^{jf3} \sin \omega^{jf}(t - t_r) \tag{6-11}$$

$$T_\alpha^{jP} = C_\alpha^{jt1} + C_\alpha^{jt2}(t - t_r) + C_\alpha^{jt3} \sin \omega^{jt}(t - t_r) . \tag{6-12}$$

In addition to the above representation, F_α^{jP} and T_α^{jP} can be input in tabular form.

6.2 Rigid-Body Environmental Disturbances

This sub-section details the equations for the rigid-body environmental disturbances. The format herein differs from that of the body of the report in that an original document is included here in its entirety. The reason for this inclusion is simply that the coding of the program proceeded from the equations contained herein and therefore it is felt that its inclusion would be preferable to a complete reordering of the equations and updating of the notation.

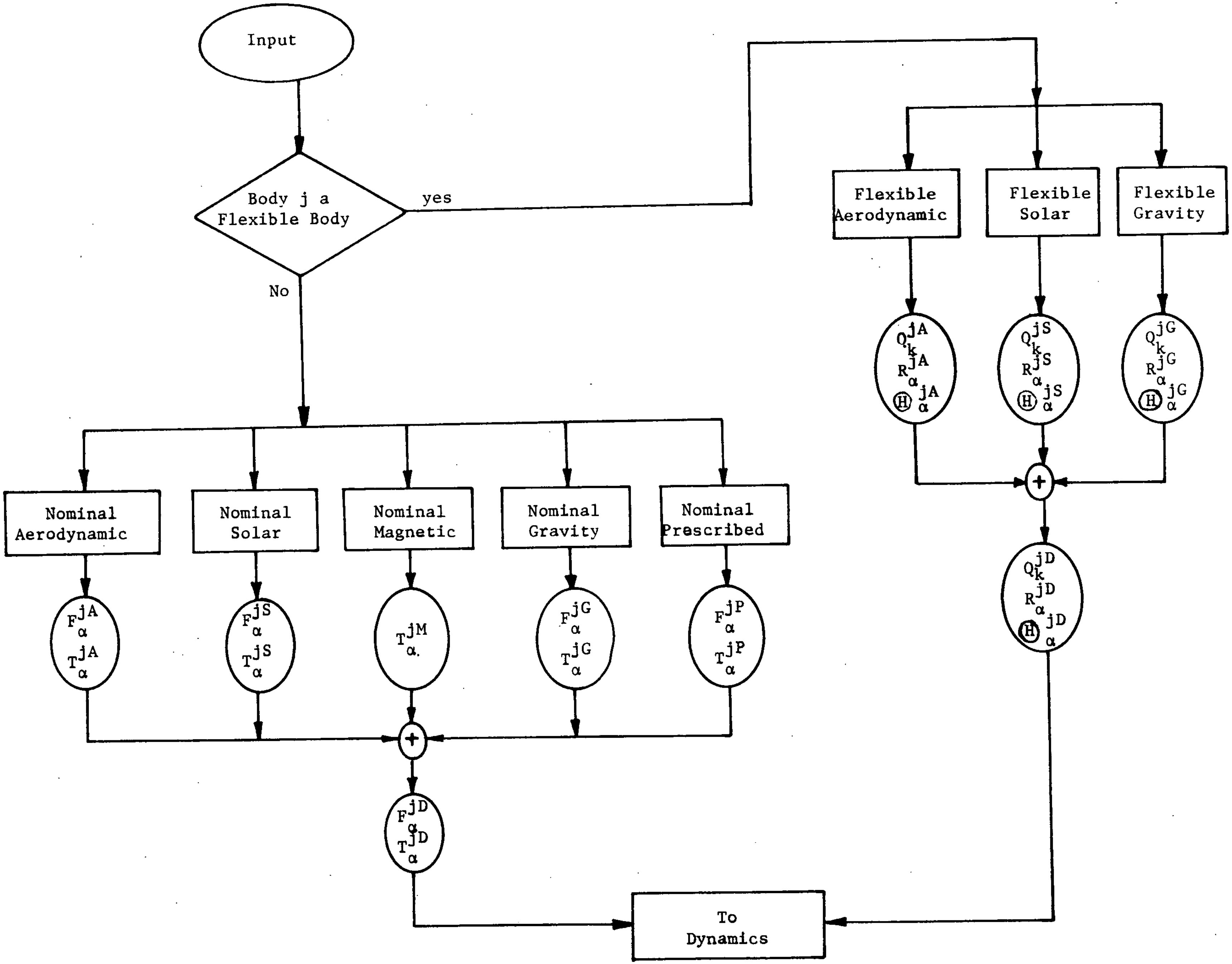


Figure 6.1. Disturban ce Subroutine Overview

1. INTRODUCTION

In what follows, forces and torques (*), due to environmental disturbances resulting from aerodynamic (**) and solar radiation pressure, gravitational attraction, and magnetic interaction, acting upon the rigid bodies of the spacecraft model are presented.

To this end and in view of the fact that the magnitudes of the forces and torques resulting from aerodynamic and solar radiation pressure are dependent on the shape of the surfaces over which they act, four basic shapes are provided for use in their computation. These consist of: a sphere, flat plate, circular cylinder, and a rectangular parallelepiped. In contrast, the gravitational interactions are dependent on the mass distribution of the bodies and their relative positions, whereas the magnetically induced forces and torques are only dependent on the positions and orientations of the bodies. Therefore, the latter two disturbances can be applied to an arbitrarily shaped body.

In the next section the force and torque (moment) equations for the four basic allowable shapes are detailed. The equations are expressed in terms of a unit flow vector $\underline{\delta}^f$, a force P^f and appropriate constants G^{jf} , H^{jf} . For solar disturbances, $\underline{\delta}^f$ becomes a unit vector $\underline{\delta}^s$ directed along a line from the sun toward the earth, P^f becomes the solar radiation pressure P^s , and G^{jf} , H^{jf} become functions of the solar reflectivity coefficient ν^j . For aerodynamic disturbances, $\underline{\delta}^f$ becomes a unit vector $\underline{\delta}^a$ directed along the negative of the spacecraft's velocity vector, P^f becomes twice the dynamic pressure P^a , and G^{jf} , H^{jf} become functions of the so-called aerodynamic reflection coefficient σ^j . More specific values of P^f , $\underline{\delta}^f$, G^{jf} and H^{jf} are given in Section 3.

(*) These forces and torques will contribute to the vectors F^{je} and T^{je} , defining the externally applied forces and torques, of Section IV.

(**) The aerodynamic interaction implied here is that corresponding to free molecular flow.

Also included in this section are the gravitational and magnetically induced forces and torques. Section 4 contains a detailed input/output specification.

In this appendix, matrix notation is used instead of the index notation employed throughout the body of the report. Given the following notational clarifications, this use of matrix form should not present difficulties in interpretation:

(i) The vector $\bar{f}$ is represented as a column matrix by $\{f\}_j$ where j denotes the coordinate system in which the components of $\bar{f}$ are given: i.e.,

$$\{f\}_j = \begin{Bmatrix} \bar{f} \cdot \underline{e}_1^j \\ \bar{f} \cdot \underline{e}_2^j \\ \bar{f} \cdot \underline{e}_3^j \end{Bmatrix}$$

j equal to r implies components in the orbital reference coordinate system (x_1^r, x_2^r, x_3^r), while $j = e$ implies components in the inertial reference system (x_1^e, x_2^e, x_3^e).

(ii) A $\hat{}$ above a symbol denotes an outer product matrix: i.e.,

$$\{\hat{\omega}^r\}_r = \begin{bmatrix} 0 & -\bar{\omega}^r \cdot \underline{e}_3^r & \bar{\omega}^r \cdot \underline{e}_2^r \\ \bar{\omega}^r \cdot \underline{e}_3^r & 0 & -\bar{\omega}^r \cdot \underline{e}_1^r \\ -\bar{\omega}^r \cdot \underline{e}_2^r & \bar{\omega}^r \cdot \underline{e}_1^r & 0 \end{bmatrix}$$

(iii) A superscript T on a matrix denotes its transpose.

2. GENERAL FORCE AND MOMENT EQUATIONS

2.1 Forces and Moments on a Sphere

If Body j is modeled as a sphere, then its surface is defined by the scalar radius R^{js}.

The total force on the sphere is given by

$$\left\{F^{jf}\right\}_j = \pi \, {R^{js}}^2 \, P^f \, [\tfrac{1}{2} H^{jf} + G^{jf}] \left\{\delta^f\right\}_j \tag{1}$$

and the total moment about the sphere's mass center is

$$\left\{T^{jf}\right\}_j = \left\{\hat{L}^{jp}\right\}_j \left\{F^{jf}\right\}_j \tag{2}$$

Where $\overline{L}^{jp}$ is the vector position of the geometric center of the sphere as measured from its mass center

2.2 Forces and Moments on a Flat Plate

If Body j is modeled as a flat plate, then its surface is defined by its scalar area A^j and by a unit normal vector $\underline{e}^{jf}$ as shown in Figure 1. The plate need not be rectangular.

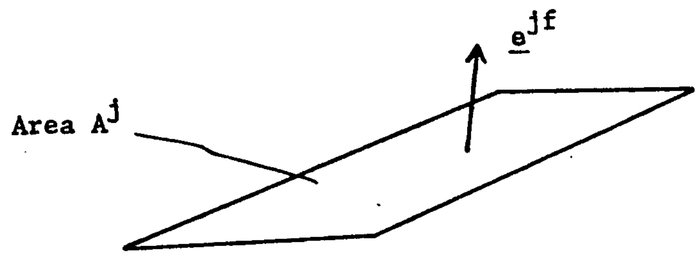


Figure 1. Surface Specification for a Flat Plate

The total force on the plate is given by

$$\left\{F^{jf}\right\}_j = P^f A^j \mid \cos \eta^j \mid \left[H^{jf} \cos \eta^j \left\{e^{jf}\right\}_j + G^{jf} \left\{\delta^f\right\}_j\right] \qquad (3)$$

where

$$\cos \eta^j = \left\{e^{jf}\right\}_j^T \left\{\delta^f\right\}_j \; . \qquad (4)$$

The total moment about the plate's mass center is

$$\left\{T^{jf}\right\}_j = \left\{\hat{L}^{jp}\right\}_j \left\{F^{jf}\right\}_j \qquad (5)$$

where $\overrightarrow{L}^{jp}$ is the vector position of the geometric center of the plate as measured from its mass center.

2.3 Forces and Moments on a Circular Cylinder

If Body j is modeled as a right circular cylinder, then its surface is defined by its scalar radius and length, R^{jc} and h^{jc} respectively and by a unit vector directed along its axis, $\underline{e}^{jf}$, as shown in Figure 2.

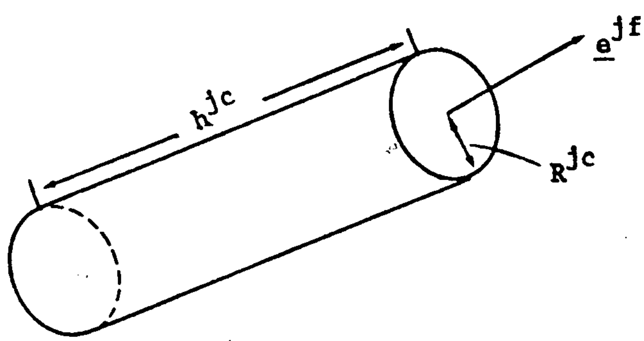


Figure 2. Surface Specification for a Cylinder

The total force on the cylinder is given by

$$\left\{F^{jf}\right\}_j = P^f R^{jc}\left[h^{jc}\sin\eta^j\,(4/3\;H^{js}+2G^{js}) + \pi R^{jc}\left|\cos\eta^j\right| G^{je}\right]\left\{\delta^f\right\}_j \tag{6}$$

$$+\;P^f R^{jc}\cos\eta^j\left[-\,4/3\;H^{js}h^{jc}\sin\eta^j + \pi R^{jc}\left|\cos\eta^j\right| H^{je}\right]\left\{e^{jf}\right\}_j$$

and the total moment about the cylinder's mass center is given by

$$\left\{T^{jf}\right\}_j = \frac{1}{2}\,\pi R^{jc^2}\,h^{jc}\,P^f\cos\eta^j\,(G^{js}-G^{je})\left\{\hat{e}^{jf}\right\}_j\left\{\delta^f\right\}_j \tag{7}$$

$$+\left\{\hat{L}^{jp}\right\}_j\left\{F^{jf}\right\}_j$$

where $\overrightarrow{L^{jp}}$ is the vector position of the geometric center of the cylinder as measured from its mass center.

In the above expressions, G^{js} and H^{js} pertain to the sides of the cylinder while G^{je} and H^{je} pertain to the ends of the cylinder and

$$\cos\eta^j = \left\{e^{jf}\right\}_j^T\left\{\delta^f\right\}_j \tag{8}$$

$$\sin\eta^j = +\sqrt{1-\cos^2\eta^j}$$

2.4 Forces and Moments on a Rectangular Parallelepiped

If Body j is modeled as a rectangular parallelepiped, then its surface is defined by three unit vectors ($\underline{e}^{j12}$, $\underline{e}^{j13}$, $\underline{e}^{j23}$) normal to any three nonparallel sides, and the lengths of these three sides L^{j1}, L^{j2}, L^{j3}. Orientation of the unit vectors is shown in Figure 3, where in general $\underline{e}^{j\alpha\beta}$ is normal to those sides with area $L^{j\alpha}\,L^{j\beta}$.

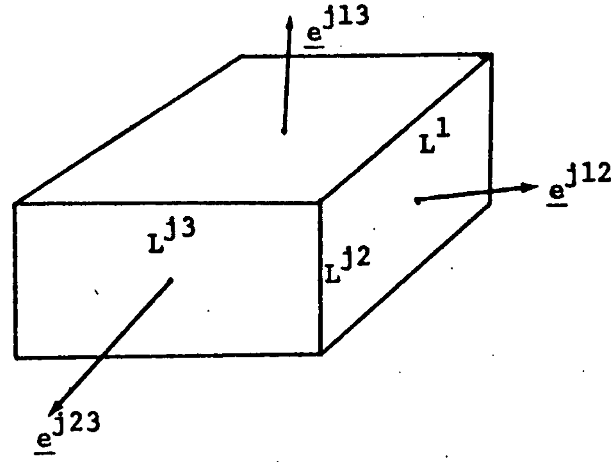


Figure 3. Surface Specification for a Parallelepiped

The total force on the parallelepiped is given by

$$\begin{aligned}\left\{F^{jf}\right\}_j = & \; P^f L^{j1} L^{j2} \left|\cos \eta^{j12}\right| \left[H^{j12} \cos \eta^{j12}\left\{e^{j12}\right\}_j + G^{j12}\left\{\delta^f\right\}_j\right] \\ & + P^f L^{j1} L^{j3} \left|\cos \eta^{j13}\right| \left[H^{j13} \cos \eta^{j13}\left\{e^{j13}\right\}_j + G^{j13}\left\{\delta^f\right\}_j\right] \\ & + P^f L^{j2} L^{j3} \left|\cos \eta^{j23}\right| \left[H^{j23} \cos \eta^{j23}\left\{e^{j23}\right\}_j + G^{j23}\left\{\delta^f\right\}_j\right]\end{aligned} \tag{9}$$

and the total moment about the parallelepiped's mass center is given by

$$\begin{aligned}\left\{T^{jf}\right\}_j = & -\frac{1}{2} L^{j1} L^{j2} L^{j3} \; P^f \left[G^{j12} \cos \eta^{j12}\left\{\hat{e}^{j12}\right\}_j + G^{j13} \cos \eta^{j13}\left\{\hat{e}^{j13}\right\}_j \right. \\ & \left. + G^{j23} \cos \eta^{j23}\left\{\hat{e}^{j23}\right\}_j\right]\left\{\delta^f\right\}_j + \left\{\hat{L}^{jP}\right\}_j \left\{F^{jf}\right\}_j\end{aligned} \tag{10}$$

where $G^{j\alpha\beta}$, $H^{j\alpha\beta}$ pertain to those sides of area $L^{j\alpha}\,L^{j\beta}$ and

$$\cos \eta^{j\alpha\beta} = \left\{e^{j\alpha\beta}\right\}_j^T \left\{\delta^f\right\}_j \tag{11}$$

Once again, $\vec{L}^{jP}$ is the vector position of the geometric center of the parallelepiped as measured from its mass center.

3. Specification of P^f, G^{jf}, H^{jf} and $\left\{\delta^f\right\}_j$

In the previous section, force and moment expressions were given for the four basic allowable shapes. These expressions contained quantities which depend on the nature of the flow, quantities which will now be defined.

3.1 Solar Radiation Pressure

If solar radiation pressure forces and moments are required, then:

I) $P^f = P^s$, the solar radiation pressure constant (P^s varies as the inverse of the square of the distance from the vehicle to the sun, and is approximately equal to 1.005×10^{-7} dyne/cm^2 in the vicinity of the earth's orbit).

II) $\left\{\delta^f\right\}_j = \left\{\delta^s\right\}_j$, a unit vector directed from the sun toward the earth. In particular,

$$\left\{\delta^s\right\}_j = \left\{A^{je}\right\} \left\{\delta^s\right\}_e \tag{13}$$

with

$$\left\{\delta^s\right\}_e = \begin{Bmatrix} -\sin \theta^s \cos 0.13\pi \\ \sin \theta^s \sin 0.13\pi \\ -\cos \theta^s \end{Bmatrix} \tag{14}$$

where

$$\theta^s = \frac{2\pi}{365.24} D^s \quad \text{[radians]} \tag{15}$$

and D^s is equal to the number of days after the autumnal equinox (it is assumed that θ^s is a constant for the period being considered and D^s is a mean value). In Equation (13), $\left\{A^{je}\right\}$ is obtained from the auxiliary calculations.

III) $$G^{jf} = 1-\nu^j \qquad H^{jf} = 2\nu^j \tag{16}$$

where ν^j is the solar reflectivity coefficient for the appropriate surface.

For the sphere and flat plate, only one value for ν^j need be specified.

For the cylinder, two distinct values for ν^j may be specified, one for the side and one for the ends of the cylinder.

For the parallelepiped, three distinct values for ν^j may be specified, one for each pair of parallel sides.

3.2 Aerodynamic Pressure

If aerodynamic pressure forces and moments are required, then:

I) $P^f = P^a$, twice the dynamic pressure. In particular,

$$P^a = \rho\, V^2 \tag{18}$$

where V^2 is the square of the magnitude of the spacecraft inertial velocity,

$$V^2 = \left\{\dot{R}^r\right\}_e^T \left\{\dot{R}^r\right\}_e \tag{19}$$

in units of $[L^2T^{-2}]$.

Here, $\left\{\dot{R}^r\right\}_e$ is obtained from the Orbital Subroutine and ρ is the atmospheric density..

II) $\left\{\delta^f\right\}_j = \left\{\delta^a\right\}_j$, a unit vector along the negative of the spacecraft's inertial velocity vector. In particular,

$$\left\{\delta^a\right\}_j = -\left\{A^{je}\right\}\left\{b^r\right\}_e \tag{20}$$

where $\left\{b^r\right\}_e$ is obtained from the Orbital Subroutine.

III) $G^{jf} = \sigma^j$

$H^{jf} = 2(1-\sigma^j)$

where σ^j is the input aerodynamic reflectivity coefficient for the appropriate surface.

For the sphere and flat plate, only one value for σ^j need be specified.

For the cylinder, two distinct values for σ^j may be specified, one for the side and one for the ends of the cylinder.

For the parallelepiped, three distinct values for σ^j may be specified, one for each pair of parallel sides.

3.3 Gravitational Force and Torque

I) The gravitational force acting on the rigid Body j is given by

$$\left\{F^{jG}\right\}_j = -\frac{\gamma m^j}{(R^r)^3}\left[\left\{A^{jr}\right\}\left\{R^j\right\}_r - 3\left\{A^{je}\right\}\left\{a^r\right\}_e\left(\left\{R^j\right\}_r^T \left\{A^{re}\right\}\left\{a^r\right\}_e\right)\right] \tag{21}$$

where γ is the input gravitational constant for the orbited body as used in the Orbital Subroutine, $\left\{a^r\right\}_e$ is obtained from the Orbital Subroutine, and m^j is the mass of Body j.

II) The gravitational torque on Body j with respect to its center of mass is

$$\left\{T^{jG}\right\}_j = \frac{3\gamma}{(R^r)^3} \left\{A^{je}\right\}\left\{\hat{a}^r\right\}_e \left\{A^{je}\right\}^T\left\{I^j\right\}_j\left\{A^{je}\right\}\left\{a^r\right\}_e \tag{22}$$

where $\left\{I^j\right\}_j$ are the components in the Body j axes of the centroidal inertia tensor of Body j.

3.4 Magnetic Torque

I) The magnetic torque on Body j resulting from its interaction with the geomagnetic field is given by the expression

$$\left\{T^{jM}\right\}_j = \left\{\hat{M}^j\right\}_j \left\{H^{jM}\right\}_j \tag{23}$$

where $\left\{H^{jM}\right\}_j$ is the magnetic field strength of the earth at Body j and $\left\{M^j\right\}_j$ is the magnetic moment of Body j. The geomagnetic field strength at Body j can be approximated by the expression(*)

$$\left\{H^{MM}\right\} = -\frac{3M^e}{(R^r)^3}\begin{Bmatrix} \sin\lambda_1 \quad \sin\lambda_2 \quad \cos\lambda_2 \\ \sin^2\lambda_2 \; - 1/3 \\ \cos\lambda_1 \quad \sin\lambda_2 \quad \cos\lambda_2 \end{Bmatrix} \tag{24}$$

Where M^e is the magnitude of the earth's magnetic dipole moment. Typically, $M^e = 8.06 \times 10^{25}$ oersted-cm^3 for the earth. Expressing equation (24) in Body j coordinates gives

$$\begin{aligned} \left\{H^{jM}\right\}_r &= \left\{A^{re}\right\}\left\{D^{eG}\right\}\left\{C^{GM}\right\}\left\{H^{MM}\right\} \\ \left\{M^{jM}\right\}_j &= \left\{A^{jr}\right\}\left\{H^{rM}\right\}_r \end{aligned} \tag{24.1}$$

II) In the above expressions

$$\lambda_2 = \sin^{-1}\left[\left\{D_2\right\}^T \left\{C^{MG}\right\}\left\{D^{Ge}\right\}\left\{A^{er}\right\}\left\{D_3\right\}\right]$$

$$\lambda_1 = \sin^{-1}\left\{\frac{\left\{D_1\right\}^T \left\{C^{MG}\right\}\left\{D^{Ge}\right\}\left\{A^{er}\right\}\left\{D_3\right\}}{\sqrt{1-\left[\left\{D_2\right\}^T \left\{C^{MG}\right\}\left\{D^{Ge}\right\}\left\{A^{er}\right\}\left\{D_3\right\}\right]^2}}\right\} \tag{26}$$

(*) Equation (24) is based on a tilted dipole model of the geomagnetic field.

$$\left\{D_1\right\} \equiv \left\{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right\} \quad ; \quad \left\{D_2\right\} \equiv \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\} \quad ; \quad \left\{D_3\right\} \equiv \left\{\begin{matrix} 0 \\ 0 \\ -1 \end{matrix}\right\} \tag{28}$$

$$\left\{D^{Ge}\right\} \equiv \begin{bmatrix} \cos\Psi^G & 0 & -\sin\Psi^G \\ 0 & 1 & 0 \\ \sin\Psi^G & 0 & \cos\Psi^G \end{bmatrix} \tag{29}$$

$$\left\{C^{MG}\right\} \equiv \begin{bmatrix} \cos\frac{11\pi}{180}\cos\frac{\pi}{9} & \sin\frac{11\pi}{180} & -\cos\frac{11\pi}{180}\sin\frac{\pi}{9} \\ -\sin\frac{11\pi}{180}\cos\frac{\pi}{9} & \cos\frac{11\pi}{180} & \sin\frac{11\pi}{180}\sin\frac{\pi}{9} \\ \sin\frac{\pi}{9} & 0 & \cos\frac{\pi}{9} \end{bmatrix} \tag{30}$$

$$\left\{C^{GM}\right\} = \left\{C^{MG}\right\}^T$$

$$\left\{D^{eG}\right\} = \left\{D^{Ge}\right\}^T$$

$$\Psi^G = \tan^{-1}\frac{\sin\theta^S\cos(0.13\pi)}{\cos\theta^S} + \frac{2\pi}{86,400}\, t^G$$

$t^G \equiv$ time in seconds after noon (GMT)

$$\left\{A^{er}\right\} \doteq \left\{A^{re}\right\}^T$$

$$\theta^S = \frac{2\pi}{365.24}\, D^S$$

4. SPECIFICATION OF INPUT AND OUTPUT QUANTITIES

The required input and output quantities will first be specified for the case of solar radiation pressure disturbances. Conventionally, (IC) denotes an input constant and [1] denotes a non-dimensional quantity.

4.1 Solar Radiation Pressure Disturbances

General Input Quantities

P^s	,	solar radiation pressure constant (IC)	$[ML^{-1}T^{-2}]$
D^s	,	mean number of days after autumnal equinox (IC)	[Days]
$\{A^{je}\}$	,	transformation matrix	[1]

4.1.1 Sphere

R^{js}	,	radius of the sphere (IC)	[L]
$\{L^{jp}\}_j$	,	position vector of the sphere's geometric center as measured from its mass center (IC)	[L]
ν^j	,	solar reflectivity coefficient (IC)	[1]

4.1.2 Flat Plate

A^j	,	area of the flat plate (IC)	$[L^2]$
$\{e^{jf}\}_j$	,	unit vector normal to flat plate (IC)	[1]
$\{L^{jp}\}_j$	,	position vector of the plate's geometric center as measured from its mass center (IC)	[L]
ν^j	,	solar reflectivity coefficient (IC)	[1]

4.1.3 Cylinder

R^{jc}	,	radius of the cylinder (IC)	[L]
h^{jc}	,	height of the cylinder (IC)	[L]
$\{e^{jf}\}_j$	,	unit vector along axis of the cylinder (IC)	[1]

$\left\{L^{jp}\right\}_j$	,	position vector of the cylinder's geometric center as measured from its mass center (IC)	[L]
ν^{js}, ν^{je}	,	solar reflectivity coefficients for sides and ends of the cylinder (IC)	[1]

4.1.4 Parallelepiped

L^{j1}, L^{j2}, L^{j3}	,	lengths of sides (IC)	[L]
$\left\{e^{j12}\right\}_j$, $\left\{e^{j13}\right\}_j$, $\left\{e^{j23}\right\}_j$	,	unit vectors normal to sides (IC)	[1]
$\left\{L^{jp}\right\}_j$	,	position vector of the parallelepiped's geometric center as measured from its mass center (IC)	[L]
ν^{j12}, ν^{j13}, ν^{j23}	,	solar reflectivity coefficients (IC)	[1]

General Output Quantities

$\left\{F^{jS}\right\}_j$	,	solar force on Body j	$[MLT^{-2}]$
$\left\{T^{jS}\right\}_j$	,	solar moment about Body j mass center	$[ML^2T^{-2}]$

4.2 Aerodynamic Pressure Disturbances

General Input Quantities

$\left\{\dot{R}^r\right\}_e$	,	velocity vector of the orbital reference frame	$[LT^{-1}]$
$\left\{A^{je}\right\}$	,	transformation matrix	[1]
$\left\{b^r\right\}_e$	,	unit vector parallel to $\dot{\bar{R}}^r$	[1]

4.2.1 Sphere

R^{js}	,	radius of the sphere (IC)	[L]
$\left\{L^{jp}\right\}_j$	,	position vector of the sphere's geometric center as measured from its mass center (IC)	[L]
σ^j	,	aerodynamic reflection coefficient (IC)	[1]

4.2.2 Flat Plate

A^j	,	area of the flat plate (IC)	$[L^2]$
$\{e^{jf}\}_j$	,	unit vector normal to flat plate (IC)	[1]
$\{L^{jp}\}_j$	,	position vector of the plate's geometric center as measured from its mass center (IC)	[L]
σ^j	,	aerodynamic reflection coefficient (IC)	[1]

4.2.3 Cylinder

R^{jc}	,	radius of the cylinder (IC)	[L]
h^{jc}	,	height of the cylinder (IC)	[L]
$\{e^{jf}\}_j$	,	unit vector along axis of the cylinder (IC)	[1]
$\{L^{jp}\}_j$	,	position vector of the cylinder's geometric center as measured from its mass center (IC)	[L]
σ^{js}, σ^{je}	,	aerodynamic reflection coefficients for sides and ends of cylinder (IC)	[1]

4.2.4 Parallelepiped

L^{j1}, L^{j2}, L^{j3}	,	lengths of sides (IC)	[L]
$\{e^{j12}\}_j$, $\{e^{j13}\}_j$, $\{e^{j23}\}_j$	,	unit vectors normal to sides	[1]
$\{L^{jp}\}_j$	,	position vector of the parallelepiped's geometric center as measured from its mass center (IC)	[L]
σ^{j12}, σ^{j13}, σ^{j23}	,	aerodynamic reflection coefficients (IC)	[1]

General Output Quantities

$\{F^{jA}\}_j$	,	aerodynamic force on Body j	$[MLT^{-2}]$
$\{T^{jA}\}_j$	,	aerodynamic moment about Body j mass center	$[ML^2T^{-2}]$

4.3 Gravitational Force and Torque

General Input Data

γ	,	gravitational constant for the orbited body (IC)	$[L^3T^{-2}]$
m^j	,	mass of Body j (IC)	[M]
R^r	,	magnitude of orbital radius vector	[L]
$\{A^{jr}\}$	,	transformation matrix	[1]
$\{R^j\}_r$	,	vector from orbital reference to Body j mass center	[L]
$\{A^{je}\}$	,	transformation matrix	[1]
$\{a^r\}_e$	,	unit vector parallel to $\overline{R}^r$	[1]
$\{A^{re}\}$	,	transformation matrix	[1]
$\{I^j\}_j$	,	Body j inertia matrix (IC)	$[ML^2]$

General Output Quantities

$\{F^{jG}\}_j$	,	gravitational force on Body j	$[MLT^{-2}]$
$\{T^{jG}\}_j$	,	gravitational torque on Body j	$[ML^2T^{-2}]$

4.4 Magnetic Torque

General Input Data

$\{M^j\}_j$	,	magnetic moment of Body j (IC)	$[ML^2T^{-2}(\text{oersted})^{-1}]$
M^e	,	magnitude of earth's magnetic field (IC)	$[L^3\ (\text{oersted})]$ (*)
$\{A^{je}\}$	.	transformation matrix	[1]

(*) $[ML^2T^{-2}(\text{oersted})^{-1}]$ is equivalent to $[L^3\text{-}(\text{oersted})]$. The latter form is included for the user's convenience, since the torque then has the proper dimensions of

$$[T^{jM}] = \frac{[ML^2T^{-2}(\text{oersted})^{-1}]}{L^3}\ [L^3\text{-}(\text{oersted})] = ML^2T^{-2}$$

$\{A^{re}\}$, transformation matrix [1]

t^G , mean time in seconds after noon (GMT)(IC) [sec]

D^s , mean time in days after autumnal equinox (IC) [days]

General Output Quantities

$\{T^{jM}\}_j$, magnetic torque on Body j [$ML^2T^{-2}(\text{oersted})^{-1}$]

$\{H^{rM}\}_r$, earth's magnetic field strength at Body; [oersted]

6.3 Flexible Body Environmental Disturbance Equations

The following equations constitute the generalized forces acting on the flexible bodies of the spacecraft model due to solar radiation pressure, aerodynamic pressure and gravity gradient effects. Derivations are contained in Appendix B.

Note here that those environmental disturbances of concern in the analysis of the Skylab vehicle have all been included (solar, aerodynamic, gravity) while magnetic disturbances are neglected since the design of the various flexible arrays (negligible dipole moment requirement) precludes the problem of magnetic interaction.

$$R_{\alpha}^{jS} = P^{s} \; A^{j} \; A_{\alpha\delta}^{jr(T)} \left\{ \nu^{j} \; B_{\sigma}^{j} \; (B_{\beta}^{j(T)} \; A_{\beta\gamma}^{je} \; S_{\gamma}^{e}) + (1 - \nu^{j}) \; A_{\delta\beta}^{je} \; S_{\beta}^{e} \right\} \left| B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; S_{\rho}^{e} \right| \qquad (6\text{-}13)$$

$$Q_{k}^{jS} = P^{s} \; A^{j} \; \Phi_{k\delta}^{j} \left\{ \nu^{j} \; B_{\delta}^{j} \; (B_{\beta}^{j(T)} \; A_{\beta\gamma}^{je} \; S_{\gamma}^{e} + (1 - \nu^{j}) \; A_{\delta\beta}^{je} \; S_{\beta}^{e} \right\} \left| B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; S_{\rho}^{e} \right| \qquad (6\text{-}14)$$

$$H_{\alpha}^{jS} = P^{s} \; A^{j} \; (\tilde{\mathring{r}}_{\alpha\beta}^{j} + \tilde{H}_{\alpha\beta}^{j3}) \left\{ \nu^{j} \; B_{\beta}^{j} \; (B_{\delta}^{j(T)} \; A_{\delta\gamma}^{je} \; b_{\gamma}^{r} + (1 - \nu^{j}) \; A_{\beta\delta}^{je} \; S_{\delta}^{e} \right\} \left| B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; b_{\rho}^{r} \right| \qquad (6\text{-}15$$

$$R_{\alpha}^{jA} = -\rho V^{2} \; A^{j} \; A_{\alpha\beta}^{jr(T)} \left\{ (1 - \sigma^{j}) \; B_{\beta}^{j} \; (B_{\delta}^{j(T)} \; A_{\delta\gamma}^{je} \; b_{\gamma}^{r}) + \sigma^{j} \; A_{\beta\delta}^{je} \; b_{\delta}^{r} \right\} \left| B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; b_{\rho}^{r} \right| \qquad (6\text{-}16)$$

$$Q_{k}^{jA} = -\rho V^{2} \; A^{j} \; \Phi_{k\beta}^{j} \left\{ (1 - \sigma^{j}) \; B_{\beta}^{j} \; (B_{\delta}^{j(T)} \; A_{\delta\gamma}^{je} \; b_{\gamma}^{r}) + \sigma^{j} \; A_{\beta\delta}^{je} \; b_{\delta}^{r} \right\} \left| B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; b_{\rho}^{r} \right| \qquad (6\text{-}17)$$

$$\textcircled{H}^{jA}_{\alpha} = -\rho V^2 \; A^j \; (\tilde{r}^{\circ j}_{\alpha\beta} + \tilde{H}^{j3}_{\alpha\beta}) \Big\{ (1 - \sigma^j) \; B^j_\beta \; (B^{j(T)}_\delta \; A^{je}_{\delta\gamma} \; b^r_\gamma) + \sigma^j \; A^{je}_{\beta\delta} \; S^e_\delta \Big\} \left| B^{j(T)}_\sigma \; A^{je}_{\sigma\rho} \; b^r_\rho \right| \tag{6-18}$$

$$R^{jG}_\alpha = -\frac{\gamma m^j}{(R^r)^3} \Big[R^j_\alpha - 3 \, A^{re}_{\alpha\beta} \; a^r_\beta \; (R^{j(T)}_\sigma \; A^{re}_{\sigma\rho} \; a^r_\rho) + A^{jr(T)}_{\alpha\beta} \; H^{j3}_\beta - 3 \, A^{re}_{\alpha\beta} \; a^r_\beta \; (a^r_\sigma \; A^{je(T)}_{\sigma\rho} \; H^{j3}_\rho) \Big] \tag{6-19}$$

$$Q^{jG}_k = -\frac{\gamma m^j}{(R^r)^3} \Big[\phi^j_{k\alpha} \; A^{jr}_{\alpha\beta} \; \eta^j_\beta + D^j_k + M^j_{k\ell} \, q^j_\ell - 3 \, c^j_{k\ell} \, q^j_\ell - 3 \; \phi^j_{k\alpha} \, A^{je}_{\alpha\beta} \; a^r_\beta \; (a^{r(T)}_\sigma \; A^{re(T)}_{\sigma\rho} \; \eta^j_\rho) - 3 \, b^j_{k\alpha} \; A^{je}_{\alpha\beta} \; a^r_\beta \Big] \tag{6-20}$$

$$\begin{aligned} \textcircled{H}^{jG}_{\alpha} = {} & \frac{3\gamma m^j}{(R^r)^3} \; A^{je}_{\alpha\beta} \; \tilde{a}^r_{\beta\delta} \; A^{je(T)}_{\delta\varepsilon} \; I^{jf}_{\varepsilon\sigma} \; A^{je}_{\sigma\rho} \; a^r_\rho \\ & - \frac{\gamma m^j}{(R^r)^3} \Big[\tilde{S}^{j6}_{\alpha\beta} \; A^{jr}_{\beta\delta} \; \eta^j_\delta - 3 \, \tilde{S}^{j6}_{\alpha\beta} \; A^{je}_{\beta\delta} \; a^r_\delta \; (\eta^{j(T)}_\sigma \; A^{re}_{\sigma\rho} \; a^r_\rho) \\ & + 3 \, A^{je}_{\alpha\beta} \; \tilde{a}^r_{\beta\delta} \; A^{je(T)}_{\delta\varepsilon} \; (b^{j(T)}_{\varepsilon k} + S^{j11(T)}_{\varepsilon k} + S^{j13(T)}_{\varepsilon k}) \; q^j_k \Big] \end{aligned} \tag{6-21}$$

where

$$\eta^j_\alpha = R^i_\alpha + A^{ir(T)}_{\alpha\beta} \; \ell^{ij}_\beta \tag{6-22}$$

$$D^j_k = \frac{1}{m^j} \int_{B^j} \bar{r}^j \cdot \bar{\phi}^j_k \; dm^j \tag{6-23}$$

$$b^j_{k\alpha} = a^{r(T)}_\varepsilon \; A^{je(T)}_{\varepsilon\beta} \; B^j_{k\beta\alpha} \tag{6-24}$$

$$B^j_{k\delta\varepsilon} = \frac{1}{m^j} \int_{B^j} \bar{r}^j \; \bar{\phi}^j_k \; dm^j \tag{6-25}$$

$$c^j_{k\ell} = a^{r(T)}_\varepsilon \; A^{je(T)}_{\varepsilon\alpha} \; C^j_{k\ell\alpha\beta} \; A^{je}_{\beta\delta} \; a^r_\delta \tag{6-26}$$

$$C^j_{k\ell\alpha\beta} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \; \bar{\phi}^j_\ell \; dm^j \tag{6-27}$$

$$S_{k\alpha}^{j11} = a_{\delta}^{r(T)} \quad A_{\delta\beta}^{je(T)} \quad B_{k\alpha\beta}^{j} \tag{6-28}$$

$$S_{k\ell\beta}^{j12} = a_{\varepsilon}^{r(T)} \quad A_{\varepsilon\alpha}^{je(T)} \quad C_{k\ell\alpha\beta}^{j} \tag{6-29}$$

$$S_{\ell\alpha}^{j13} = q_{k}^{j} \quad S_{k\ell\alpha}^{j12} \tag{6-30}$$

VII. Orbital Subroutine

The dynamic equations for a multibodied flexible spacecraft presented in the previous sections of this report are written with respect to a reference axis frame. One important choice for specification of this reference axis frame is that it lie on a user-defined Kepler orbit. The orbit routine, as specified in this section, calculates this reference orbit. The orbit equations contained herein were specified concurrent to the rigid-body environmental disturbance equations and thus are likewise included here in their original form in order to retain correspondence with the computer programming documentation. As in the case of Section 6.2, matrix notation is used herein instead of the index notation employed throughout the body of this report. (The orbit subroutine commences here with Section 2.0 following.)

2.0 COORDINATE SYSTEMS AND NOTATION

In describing the motion of the reference axis frame, two right-handed, orthogonal coordinate systems are of basic importance. These coordinate systems are defined and described below. The inertial reference frame is defined with respect to the earth as the orbited body. In case orbital motion about another celestial body is desired, an appropriate inertial reference frame must be defined.

2.1 Inertially Fixed Coordinate System

The inertially fixed coordinate system with axes (x_1^e, x_2^e, x_3^e) has its origin at the mass center of the earth and is defined by unit vectors $\underline{e}_1^e$, $\underline{e}_2^e$, $\underline{e}_3^e$ directed along the appropriate axes in a positive sense. The unit vector $\underline{e}_2^e$ is normal to the equatorial plane and positive northward, while $\underline{e}_3^e$ is directed along the autumnal equinox. The coordinate matrix for this system is given by

$$\left\{\underline{e}^e\right\} = \left\{\begin{matrix} \underline{e}_1^e \\ \underline{e}_2^e \\ \underline{e}_3^e \end{matrix}\right\} . \tag{2.1}$$

Since the coordinate system is inertially fixed, its rotational velocity vector is given by

$$\bar{\omega}^e \equiv 0 . \tag{2-2}$$

2.2 Orbital Reference Coordinate System

The orbital reference coordinate system with coordinate axes (x_1^r, x_2^r, x_3^r) is defined by unit vectors $\underline{e}_1^r$, $\underline{e}_2^r$, $\underline{e}_3^r$ such that $\underline{e}_3^r$ points toward the earth's mass center, $\underline{e}_1^r$ lies in the orbit plane and forms an acute angle with the tangential velocity vector while $\underline{e}_2^r$ is normal to the orbit plane. The coordinate matrix for this system is given by

$$\left\{\underline{e}^r\right\} = \left\{\begin{array}{c} \underline{e}_1^r \\ \underline{e}_2^r \\ \underline{e}_3^r \end{array}\right\} \tag{2-3}$$

and its angular velocity vector is denoted by

$$\overline{\omega^r} \neq 0 \quad . \tag{2-4}$$

The vector distance from the center of mass of the earth to the origin of (x_1^r, x_2^r, x_3^r) is $\overline{R^r}$.

2.3 Notation

The following common notational rules will be used in the sequel:

(i) The vector $\overline{f}$ is represented as a column matrix by $\left\{f\right\}_j$ where j denotes the coordinate system in which the components of $\overline{f}$ are given; i.e.,

$$\left\{f\right\}_j = \left\{\begin{array}{c} \overline{f} \cdot \underline{e}_1^j \\ \overline{f} \cdot \underline{e}_2^j \\ \overline{f} \cdot \underline{e}_3^j \end{array}\right\} \quad .$$

j equal to r implies components in the orbital reference coordinate system (x_1^r, x_2^r, x_3^r), while j = e implies components in the inertial reference system (x_1^e, x_2^e, x_3^e).

(ii) A ^ above a symbol denotes an outer product matrix; i.e.,

$$\left\{\hat{\omega}^{r}\right\}_{r} = \begin{bmatrix} 0 & -\bar{\omega}^{r} \cdot \underline{e}_3^{r} & \bar{\omega}^{r} \cdot \underline{e}_2^{r} \\ \bar{\omega}^{r} \cdot \underline{e}_3^{r} & 0 & -\bar{\omega}^{r} \cdot \underline{e}_1^{r} \\ -\bar{\omega}^{r} \cdot \underline{e}_2^{r} & \bar{\omega}^{r} \cdot \underline{e}_1^{r} & 0 \end{bmatrix} .$$

(iii) A superscript T on a matrix denotes its transpose.

The following sections are devoted to the determination of $\bar{R}^r$, $\bar{\omega}^r$ and their required time derivatives.

3.0 ORBIT EQUATIONS

Let us now define the orbit equations for the reference frame (x_1^r, x_2^r, x_3^r). The position of the reference frame is determined by the equation for an inverse-square central force

$$\frac{d^2\,\bar{R}^r}{dt^2} = \frac{\bar{F}_g}{M}$$

where

$$\bar{F}_g = -\frac{\gamma M}{(R^r)^3}\,\bar{R}^r ,$$

with: M = total spacecraft mass [M]
R^r = magnitude of $\bar{R}^r$ [L]
γ = gravitational constant for the orbited body (GM_b) [$L^3\,T^{-2}$]
where G is the universal gravitational constant
and M_b is the mass of the orbited body

(For the earth, $\gamma = 3.98604 \times 10^5\ \mathrm{km^3/sec^2} = 1.40766 \times 10^{16}\ \mathrm{ft^3/sec^2}$)

Thus, the position of the reference frame is determined by

$$\frac{d^2\overline{R}^r}{dt^2} = - \frac{\gamma\overline{R}^r}{(R^r)^3} \tag{3.1}$$

Equation (3.1) represents the motion of a point in an elliptical orbit about a spherical planet, Figure 3.1.

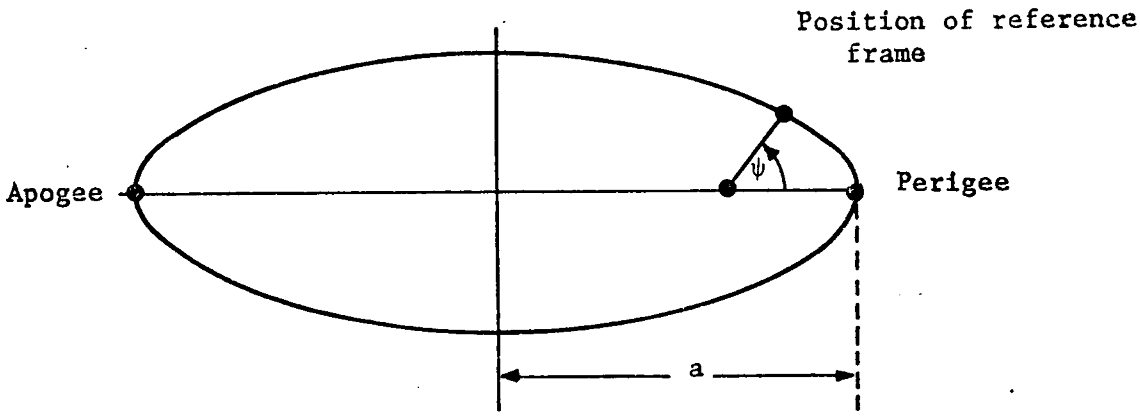


Figure 3.1 Orbit Parameters

Letting

P = orbital period [T]

ε = eccentricity of orbit [non-dimensional]

the semi-major axis of the orbit is given by

$$a = \left(\frac{\sqrt{\gamma}}{\omega_o}\right)^{2/3}, \; [L] \tag{3-2}$$

where the "mean motion" ω_o is defined by

$$\omega_o = \frac{2\pi}{P}, \; [T^{-1}] \tag{3-3}$$

with ω_o being the equivalent circular orbital frequency in radians per unit of time.

The true anomaly ψ is determined by

$$\psi = 2 \tan^{-1} \left[\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \left(\frac{E}{2}\right)\right] \tag{3-4}$$

where the eccentric anomaly E must be found as a solution of Kepler's equation

$$E - \varepsilon \sin E = \omega_o (t - t_p) . \tag{3-5}$$

Here, t_p is the time at perigee and the quantity $\omega_o(t - t_p)$ is the <u>mean anomaly.</u>

The radial distance to the reference frame, R^r, is given by

$$R^r = \frac{a\ (1-\varepsilon^2)}{1 + \varepsilon\cos\psi}. \quad [L] \tag{3-6}$$

In addition,

$$\dot{\psi} = \frac{\sqrt{a\gamma(1-\varepsilon^2)}}{R^{r2}} \quad . \quad [T^{-1}] \tag{3-7}$$

By differentiating the above relationships we find that

$$\dot{R}^r = \sqrt{\frac{\gamma}{a\ (1-\varepsilon^2)}}\ \varepsilon\ \sin\psi\ [LT^{-1}] \tag{3-8}$$

$$\ddot{R}^r = \frac{\varepsilon\gamma\cos\ \psi}{R^{r\ 2}}\ , \quad [LT^{-2}] \tag{3-9}$$

$$\ddot{\psi} = -\frac{2\ \varepsilon\gamma\sin\psi}{R^{r\ 3}} \quad . \quad [T^{-2}] \tag{3-10}$$

4.0 <u>MOTION OF ORBITAL REFERENCE SYSTEM</u>

The inertial reference system (x_1^e, x_2^e, x_3^e), the orbital reference system (x_1^r, x_2^r, x_3^r), the transformation between them $\{A^{re}\}$ and the angular velocity vector $\overline{\omega}^r$ of the orbital reference frame are independent of the relative motic of the bodies of the spacecraft and can be written as functions of the orbit parameters and time. We now determine these functional relationships.

4.1 <u>The Inertial Reference Axes</u>

The inertial reference axes have their origin at the center of mass of the earth.

The x_2^e axis is normal to the equatorial plane and is positive in the northward direction; the x_3^e axis lies along the autumnal equinox.

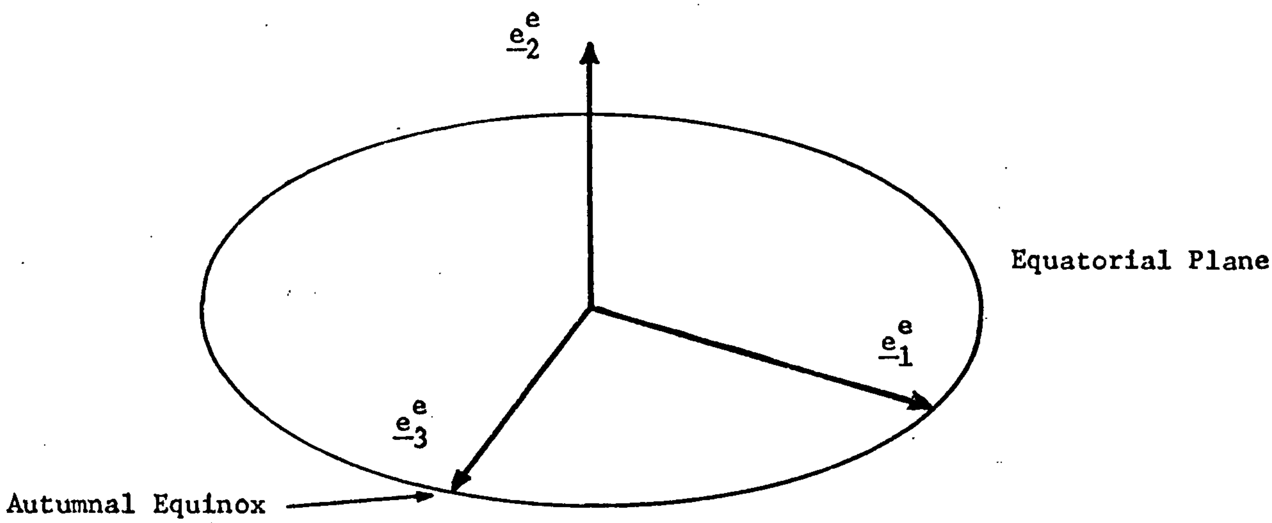


Figure 4.1. Inertial Reference System

The coordinate matrix of the inertial axes is given by (2-1) as

$$\left\{\underline{e}^e\right\} = \left\{\begin{matrix} \underline{e}_1^e \\ \underline{e}_2^e \\ \underline{e}_3^e \end{matrix}\right\}.$$

4.2 The Orbital Reference Axes

A rotation of the inertial axes through an angle β about the $\underline{e}_2^e$ axis produces the $\left\{\underline{e}^\beta\right\}$ system in which $\underline{e}_3^\beta$ lies along the line of nodes (intersection of the orbit plane and the equatorial plane, Figure 4.2) with $\underline{e}_3^\beta$ directed toward the ascending node.

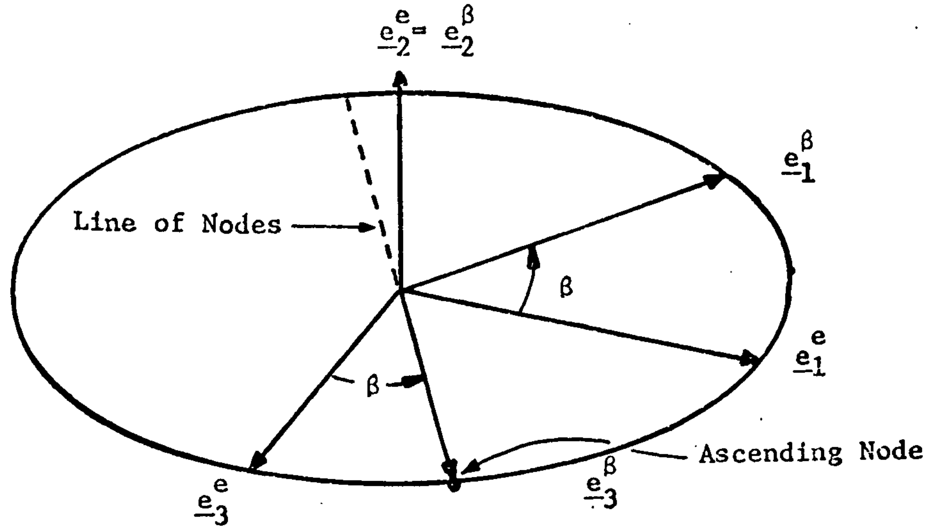


Figure 4.2. Line of Nodes

With $\left\{\underline{e}^\beta\right\} = \begin{Bmatrix} \underline{e}_1^\beta \\ \underline{e}_2^\beta \\ \underline{e}_3^\beta \end{Bmatrix}$, let $\left\{A^{\beta e}\right\}$ be the transformation matrix which transforms the $\left\{\underline{e}^e\right\}$ system into the $\left\{\underline{e}^\beta\right\}$ system such that

$$\left\{\underline{e}^\beta\right\} = \left\{A^{\beta e}\right\}\left\{\underline{e}^e\right\}$$

where

$$\left\{A^{\beta e}\right\} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \qquad (4\text{-}1)$$

and

$$\left\{\dot{A}^{\beta e}\right\} = \dot{\beta} \begin{bmatrix} -\sin\beta & 0 & -\cos\beta \\ 0 & 0 & 0 \\ \cos\beta & 0 & -\sin\beta \end{bmatrix} . \tag{4-2}$$

Rotation of the $\left\{\underline{e}^{\beta}\right\}$ system, Figure 4.2, about $\underline{e}_3^{\beta}$ through an angle ξ, the <u>orbit inclination angle</u>, produces the $\left\{\underline{e}^{\xi}\right\}$ system, Figure 4.3, where $\underline{e}_1^{\xi}$ lies in the orbit plane.

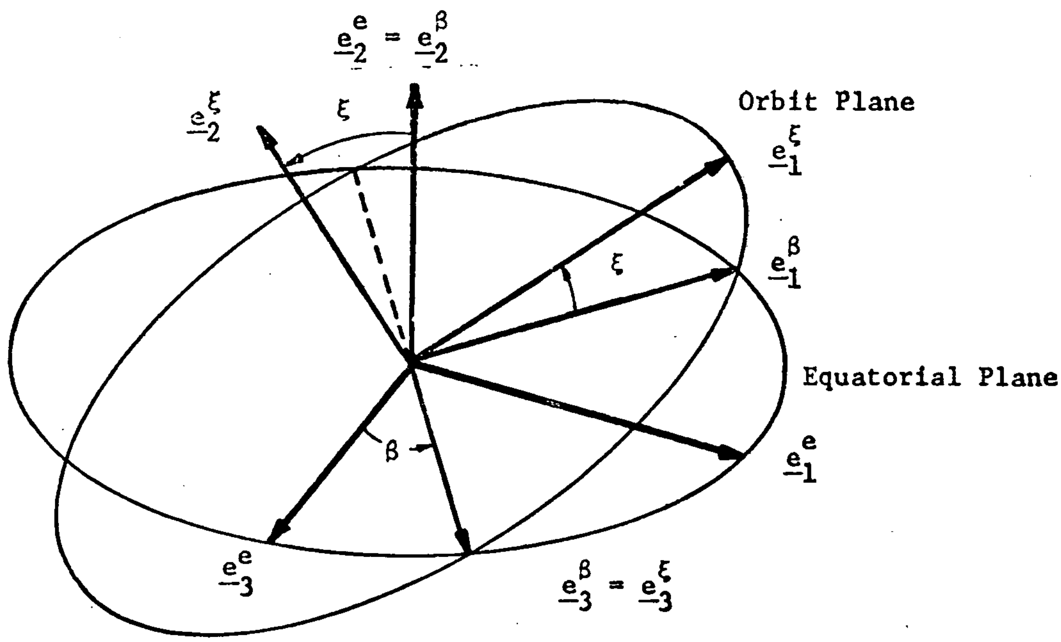


Figure 4.3. Orbit Inclination

With $\left\{\underline{e}^{\xi}\right\} = \begin{Bmatrix} \underline{e}_1^{\xi} \\ \underline{e}_2^{\xi} \\ \underline{e}_3^{\xi} \end{Bmatrix}$, let $\left\{A^{\xi\beta}\right\}$ be the transformation matrix which transforms the $\left\{\underline{e}^{\beta}\right\}$ system into the $\left\{\underline{e}^{\xi}\right\}$ system such that

$$\left\{\underline{e}^{\xi}\right\} = \left\{A^{\xi\beta}\right\}\left\{\underline{e}^{\beta}\right\}$$

where

$$\left\{A^{\xi\beta}\right\} = \begin{bmatrix} \cos\xi & \sin\xi & 0 \\ -\sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4-3}$$

Since ξ is a constant, $\dot{\xi} = 0$ and

$$\left\{\dot{A}^{\xi\beta}\right\} = 0. \tag{4-4}$$

Rotation of the $\left\{\underline{e}^{\xi}\right\}$ system, Figure 4.3, about $\underline{e}_2^{\xi}$ through an angle α produces the $\left\{\underline{e}^{\alpha}\right\}$ system, Figure 4.4, where $\underline{e}_3^{\alpha}$ passes through the orbital position of the reference frame.

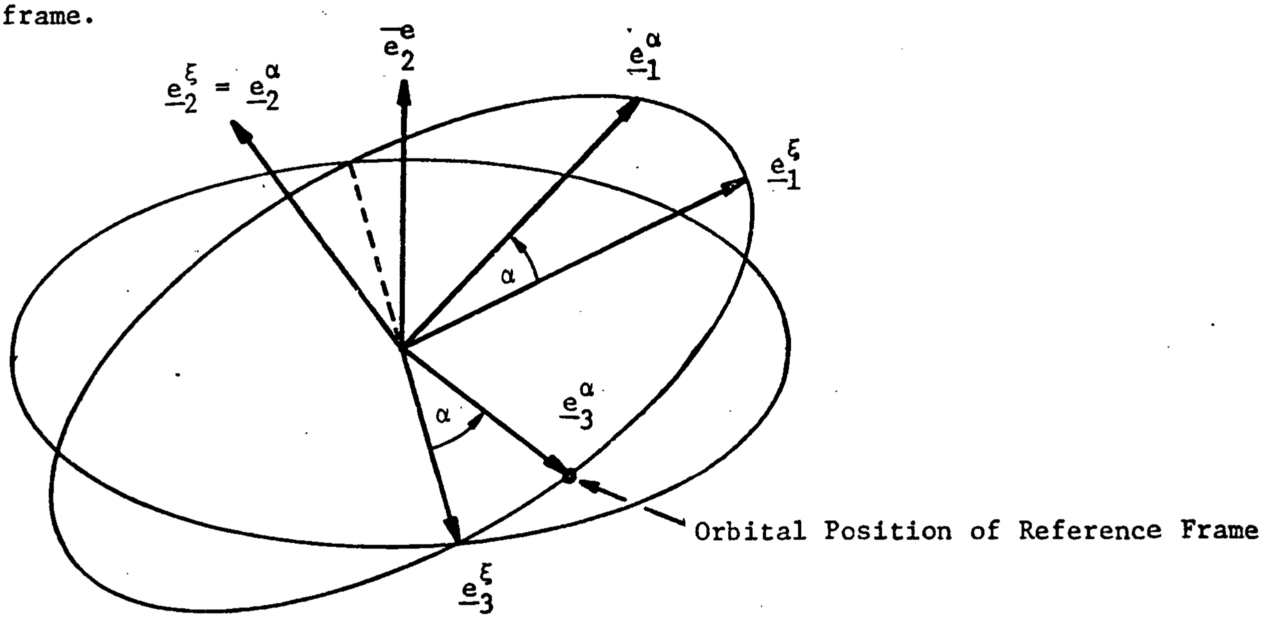


Figure 4.4 Orbit Angles

With $\left\{\underline{e}^{\alpha}\right\} = \begin{Bmatrix} \underline{e}_1^{\alpha} \\ \underline{e}_2^{\alpha} \\ \underline{e}_3^{\alpha} \end{Bmatrix}$, let $\left\{A^{\alpha\xi}\right\}$ be the transformation matrix which transforms the $\left\{\underline{e}^{\xi}\right\}$ system into the $\left\{\underline{e}^{\alpha}\right\}$ system such that

$$\left\{\underline{e}^{\alpha}\right\} = \left\{A^{\alpha\xi}\right\}\left\{\underline{e}^{\xi}\right\}$$

where

$$\left\{A^{\alpha\xi}\right\} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{4-5}$$

and

$$\left\{\dot{A}^{\alpha\xi}\right\} = \dot{\alpha} \begin{bmatrix} -\sin\alpha & 0 & -\cos\alpha \\ 0 & 0 & 0 \\ \cos\alpha & 0 & -\sin\alpha \end{bmatrix} \tag{4-6}$$

The orbital reference system (x_1^r, x_2^r, x_3^r) with unit vectors $\underline{e}_1^r$, $\underline{e}_2^r$, $\underline{e}_3^r$ has its origin at the orbital reference point, with $\underline{e}_3^r$ pointing toward nadir, $\underline{e}_1^r$ in the orbit plane forming an acute angle with the velocity vector and $\underline{e}_2^r$ normal to the orbit plane, Figure 4.5.

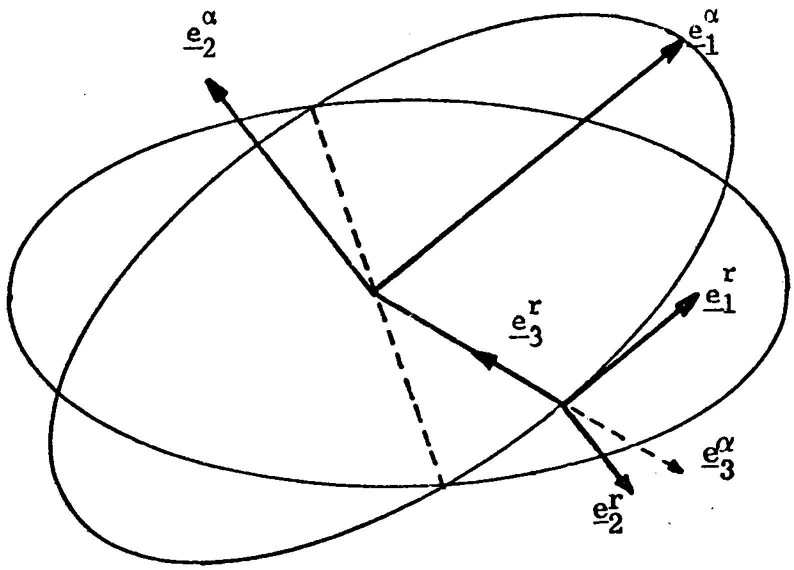


Figure 4.5. Orbital Reference System

Thus,

$$\underline{e}_1^r = \underline{e}_1^\alpha, \ \underline{e}_2^r = -\underline{e}_2^\alpha, \ \underline{e}_3^r = -\underline{e}_3^\alpha \ .$$

Letting $\left\{\underline{e}^r\right\} = \begin{Bmatrix} \underline{e}_1^r \\ \underline{e}_2^r \\ \underline{e}_3^r \end{Bmatrix}$ and letting $\left\{A^{r\alpha}\right\}$ be the transformation matrix

which transforms the $\left\{\underline{e}^\alpha\right\}$ system into the $\left\{\underline{e}^r\right\}$ system, we have

$$\left\{\underline{e}^r\right\} = \left\{A^{r\alpha}\right\}\left\{\underline{e}^\alpha\right\}$$

where

$$\left\{A^{r\alpha}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4-7}$$

and $\left\{\dot{A}^{r\alpha}\right\} = 0$.

Combining the previous equations,

$$\left\{\underline{e}^{r}\right\} = \left\{A^{r\alpha}\right\}\left\{A^{\alpha\xi}\right\}\left\{\underline{e}^{\xi}\right\}$$

or, letting $\left\{A^{r\xi}\right\} = \left\{A^{r\alpha}\right\}\left\{A^{\alpha\xi}\right\}$,

$$\left\{\underline{e}^{r}\right\} = \left\{A^{r\xi}\right\}\left\{\underline{e}^{\xi}\right\} \tag{4-8}$$

where

$$\left\{A^{r\xi}\right\} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & -1 & 0 \\ -\sin\alpha & 0 & -\cos\alpha \end{bmatrix} \tag{4-9}$$

and

$$\left\{\dot{A}^{r\xi}\right\} = \dot{\alpha} \begin{bmatrix} -\sin\alpha & 0 & -\cos\alpha \\ 0 & 0 & 0 \\ -\cos\alpha & 0 & \sin\alpha \end{bmatrix} . \tag{4-10}$$

Similarly,

$$\left\{\underline{e}^{r}\right\} = \left\{A^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{A^{\beta e}\right\}\left\{\underline{e}^{e}\right\}$$

or letting, $\left\{A^{re}\right\} = \left\{A^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{A^{\beta e}\right\}$,

$$\left\{\underline{e}^{r}\right\} = \left\{A^{re}\right\}\left\{\underline{e}^{e}\right\} \tag{4-11}$$

where

$$\left\{A^{re}\right\} = \begin{bmatrix} (\cos\alpha\cos\xi\cos\beta - \sin\alpha\sin\beta) & (\cos\alpha\sin\xi) & (-\cos\alpha\cos\xi\sin\beta - \sin\alpha\cos\beta) \\ (\sin\xi\cos\beta) & (-\cos\xi) & (-\sin\xi\sin\beta) \\ (-\sin\alpha\cos\beta\cos\xi - \cos\alpha\sin\beta) & (-\sin\alpha\sin\xi) & (\sin\alpha\cos\xi\sin\beta - \cos\alpha\cos\beta) \end{bmatrix} \quad (4\text{-}12)$$

Thus, $\left\{A^{re}\right\}$ is a function of the three orbital parameters β, ξ and α. The angle α has been defined as the orbital position of the reference frame with respect to the ascending node while ψ has previously been defined as the true anomaly or as the orbital reference position with respect to perigee. Therefore, if α_p is the orbit angle between the ascending node and perigee, then

$$\alpha = \alpha_p + \psi$$

and from Equation (2-7), (2-10):

$$\dot{\alpha} = \dot{\psi} = \frac{\sqrt{a\gamma\,(1-\varepsilon^2)}}{{R^r}^2}\,, \quad [T^{-1}] \quad (4\text{-}13)$$

$$\ddot{\alpha} = \ddot{\psi} = \frac{-2\,\varepsilon\gamma\,\sin\psi}{{R^r}^3}\,. \quad [T^{-2}] \quad (4\text{-}14)$$

The orbital inclination, ξ , is an input constant. The nodal regression rate $\dot{\beta}$ due to the orbited body's oblateness is given by

$$\dot{\beta} = -\frac{3J_2}{2(1-\varepsilon^2)^2}\left(\frac{R_b}{a}\right)^2 \omega_o \cos\xi \quad [T^{-1}] \tag{4-15}$$

where R_b is the radius of the orbited body and J_2 is the second harmonic of the orbited body's gravitational potential. For the earth,

$$R_b = 6.378 \times 10^3 \text{ km} = 2.093 \times 10^7 \text{ ft}$$

$$J_2 = 1082.28 \pm 0.3 \times 10^{-6}$$

(The above expression for $\dot{\beta}$ as well as the value for J_2 of the earth are found in the Reference below (*). If β_o is the value of β at time of the first nodal crossing, t_n , then the parameter β is given by

$$\beta = \beta_o + \dot{\beta}(t-t_n). \tag{4-16}$$

Since the matrix $\left\{\dot{A}^{\xi\beta}\right\}$ is zero (Equation 4-4), differentiating the relation for $\left\{A^{re}\right\}$ gives

$$\left\{\dot{A}^{re}\right\} = \left\{\dot{A}^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{A^{\beta e}\right\} + \left\{A^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{\dot{A}^{\beta e}\right\} . \tag{4-17}$$

But, it can be shown that

$$\left\{\dot{A}^{re}\right\} = -\left\{\hat{\omega}^{r}\right\}_r\left\{A^{re}\right\} . \tag{4-18}$$

Rewriting Equation (4-17), we find that

$$\left\{\dot{A}^{re}\right\} = \left\{\dot{A}^{r\xi}\right\}\left\{A^{r\xi}\right\}^T\left\{A^{re}\right\} + \left\{A^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{\dot{A}^{\beta e}\right\}\left\{A^{\beta e}\right\}^T\left\{A^{\xi\beta}\right\}^T\left\{A^{r\xi}\right\}^T\left\{A^{re}\right\}$$

so that

$$\left\{\hat{\omega}^{r}\right\}_r = -\left\{\dot{A}^{r\xi}\right\}\left\{A^{r\xi}\right\}^T - \left\{A^{r\xi}\right\}\left\{A^{\xi\beta}\right\}\left\{\dot{A}^{\beta e}\right\}\left\{A^{\beta e}\right\}^T\left\{A^{\xi\beta}\right\}^T\left\{A^{r\xi}\right\}^T \tag{4-19}$$

(*)Escobal, P. R., Methods of Orbit Determination, John Wiley and Sons, Inc., New York, New York, 1965.

Expanding the above expression,

$$\left\{\hat{\omega}^r\right\}_r = \left\{\hat{m}^r\right\}_r + \left\{\hat{n}^r\right\}_r$$

where

$$\left\{m^r\right\}_r = \dot{\beta} \begin{Bmatrix} \sin\xi\cos\alpha \\ -\cos\xi \\ -\sin\xi\sin\alpha \end{Bmatrix}$$

$$\left\{n^r\right\}_r = \dot{\alpha} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} .$$

Therefore,

$$\left\{\omega^r\right\}_r = \begin{Bmatrix} \dot{\beta}\sin\xi\cos\alpha \\ -\dot{\beta}\cos\xi-\dot{\alpha} \\ -\dot{\beta}\sin\xi\sin\alpha \end{Bmatrix} \tag{4.20}$$

and, since ξ is constant,

$$\left\{\dot{\omega}^r\right\}_r = \begin{Bmatrix} -\dot{\beta}\dot{\alpha}\sin\xi\sin\alpha \\ -\ddot{\alpha} \\ -\dot{\beta}\dot{\alpha}\sin\xi\cos\alpha \end{Bmatrix} . \tag{4-21}$$

Finally, letting

$$\left\{S^r\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4-22}$$

the following quantities are defined:

$$\left\{R^r\right\}_e = -R^r \left\{A^{re}\right\}^T \left\{S^r\right\} \tag{4-23}$$

$$\left\{a^r\right\}_e = \left\{R^r\right\}_e / R^r \tag{4-24}$$

$$\left\{\dot{R}^r\right\}_e = -\left\{A^{re}\right\}^T \left(\dot{R}^r \left\{S^r\right\} + R^r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}\right) \tag{4-25}$$

$$V^2 = \left\{\dot{R}^r\right\}_e^T \left\{\dot{R}^r\right\}_e \tag{4-25.1}$$

$$b^r{}_e = \left\{\dot{R}^r\right\}_3 / V \tag{4-26}$$

$$\begin{aligned} \ddot{R}^r{}_e = & -\left\{A^{re}\right\}^T \left(\ddot{R}^r \left\{S^r\right\} + 2\,\dot{R}^r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}\right. \\ & \left. + R^r \left\{\hat{\dot{\omega}}^r\right\}_r \left\{S^r\right\} + R^r \left\{\hat{\omega}^r\right\}_r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}\right) \end{aligned} \tag{4-27}$$

5.0 CONCLUSIONS

We have here documented the determination of those orbital parameters necessary to specify the reference axis frame in the given Kepler orbit. In particular, the requisite input parameters are simply:

α_o = orbit angle from line of nodes at time zero. [radian]

α_p = orbit angle between ascending node and perigee. [radian]

β_o = angle between autumnal equinox and the ascending line of nodes measured in the equatorial plane at time t_n. [radian]

P = orbital period [T]

ε = orbital eccentricity. [-]

ξ = orbit inclination with respect to equatorial plane. [radians]

R_b = radius of the orbited body [L]

γ = gravitational constant for the orbited body $[L^3\ T^{-2}]$

J_2 = second harmonic of the orbited body's gravitational potential [-]

t_n = time of first nodal crossing $(-P<t_n\leq 0)$ [T]

The output parameters are:

R^r = magnitude of orbital radius vector $\bar{R}^r$ [L]

V = magnitude of orbital velocity vector $\dot{\bar{R}}^r$ $[LT^{-1}]$

$\left\{\omega^r\right\}_r$ = inertial angular velocity of orbital reference frame $[T^{-1}]$

$\left\{\dot{\omega}^r\right\}_r$ = inertial angular acceleration of orbital reference frame $[T^{-2}]$

$\left\{A^{re}\right\}$ = transformation matrix from inertial to orbital reference axes [-]

$\left\{R^r\right\}_e$ = column vector representation of $\bar{R}^r$. [L]

$\left\{a^r\right\}_e$ = unit vector along $\bar{R}^r$. [-]

$\left\{\dot{R}^r\right\}_e$ = column vector representation of $\dot{\bar{R}}^r$. $[LT^{-1}]$

$\left\{b^r\right\}_e$ = unit vector along $\dot{\bar{R}}^r$. [-]

$\left\{\ddot{R}^r\right\}_e$ = column vector representation of $\ddot{\bar{R}}^r$. $[LT^{-2}]$

6.0 EXISTING COMPUTER PROGRAMS

Since a digital orbital routine identical to the one presented in this report has been successfully coded in the existing TRW Generalized Spacecraft Simulation (GSS) program, this routine is included here for reference purposes. (Note change in expression for $\dot{\beta}$ and inclusion of (59)-(65).)

6.1 Subroutine SETØRB

This subroutine initializes the orbit parameters and computes the constant portion of the orbital computations. The initial eccentric anomaly (EK) is computed and used as the starting value for the first iteration of Kepler's equation. TPER, the time at which the spacecraft last crossed perigee is computed. The semi-major axis of the orbit (AXIS) and a related constant (AESQ) are computed from which the nodal regression rate (BDØT) is computed.

SUBRØUTINE SETØRB

$$\alpha = \alpha_o$$

$$\psi = \alpha_o - \alpha_p$$

$$E_k = 2.\ T_{an}^{-1} \left(\sqrt{\frac{1.-\varepsilon}{1.+\varepsilon}} \quad \frac{\sin(\psi)}{1.+\cos(\psi)} \right)$$

$$* \ TPER = \frac{-(E_k - \varepsilon \sin(E_k))P}{2\pi}$$

$$\beta = \beta_o$$

$$* \ a = (P\sqrt{\gamma}/2\pi)^{2/3}$$

$$AESQ = a(1.- \varepsilon^2)$$

$$* \ \dot{\beta} = - \frac{3.\ J_2}{2.(1-\varepsilon^2)^2} \left(\frac{R_b}{a}\right)^2 \frac{2\pi \cos(\xi)}{P}$$

6.2 Subroutine ØRBIT

This routine computes the orbit angle ALPH (the true anomaly measured at the center of the earth between perigee and the origin of the reference frame) at time T by solving Kepler's equation using Newton's method of iteration. Each iteration is begun by assuming the previous value of ALPH as a starting value. A maximum of ten iterations is allowed, the iterations ending when the difference between two successive values is less than 10^{-5}. If this condition is not met within ten iterations, the initial value is changed, a message is printed out and the program begins the next case.

All orbit related parameters are computed within this routine, including the orbit radius (RR), the radial velocity (RRDØT) and acceleration (RRDDT), the orbital angular velocity (PSDØT) and acceleration (PSDDT), the angular velocity of the orbital reference frame relative to inertial axes (ØMRRC) and its derivative (ØMRRD), the gravitational acceleration at the reference frame origin (CØNG), and the angle between the autumnal equinox and the line of nodes (BETA).

SUBRØUTINE ØRBIT

* (1) $\text{ECOUT} = 2\pi\ (T - \text{TPER})/P$

(5) $\text{SLØPE} = 1. - \varepsilon\cos\ (E_k)$

(7) $\text{STEP} = \dfrac{E_k - \varepsilon\sin\ (E_k) - \text{ECOUT}}{\text{SLØPE}}$

(8) $E_k = E_k - \text{STEP}$

(22) $E_k = E_k\ (1. - 0.1\ \text{KP})$

where KP = Number of Restarts

$$(36) \quad \psi = 2.\ T_{an}^{-1}\left[\sqrt{\frac{1.+\varepsilon}{1.-\varepsilon}}\ T_{an}\left(\frac{E_k}{2.}\right)\right]$$

$$(44) \quad ECPS = 1.\ +\epsilon\cos\ (\psi)$$

$$(45) \quad R^r = AESQ/ECPS$$

$$* \ (46) \quad C\emptyset NG = \gamma/R^{r^2}$$

$$* \ (47) \quad \dot{R}^r = \varepsilon\ \sin(\psi)\sqrt{\gamma/AESQ}$$

$$(48) \quad \ddot{R}^r = C\emptyset NG\ \ \varepsilon\ \ \cos(\psi)$$

$$(49) \quad \dot{\psi} = \sqrt{\gamma\ AESQ}\ \ /R^{r^2}$$

$$(50) \quad \ddot{\psi} = -2.\gamma\ \ \varepsilon\ \ \sin(\psi)/R^{r^3}$$

$$(53) \quad \beta = \beta_o + \dot{\beta}(t-t_n)$$

$$(57) \quad \left\{\omega^r\right\}_r = \begin{Bmatrix} \dot{\beta}\sin(\xi)\ \cos(\alpha) \\ -\dot{\beta}\ \cos(\xi)\ -\dot{\psi} \\ -\dot{\beta}\ \sin(\xi)\ \sin(\alpha) \end{Bmatrix}$$

$$(58) \quad \left\{\dot{\omega}^r\right\}_r = \begin{Bmatrix} -\dot{\beta}\dot{\psi}\sin(\xi)\ \sin(\alpha) \\ -\ddot{\psi} \\ -\dot{\beta}\ \dot{\psi}\sin(\xi)\cos\ (\alpha) \end{Bmatrix}$$

$$(59) \quad \left\{A^{re}\right\} = \begin{bmatrix} (\cos\alpha\cos\xi\cos\beta-\sin\alpha\sin\beta) & (\cos\alpha\sin\xi) & (-\cos\alpha\cos\xi\sin\beta-\sin\alpha\cos\beta) \\ (\quad \sin\xi\cos\beta \quad) & (\ -\cos\xi\ ) & (\quad -\sin\xi\sin\beta \quad) \\ (-\sin\alpha\cos\beta\cos\xi-\cos\alpha\sin\beta) & (-\sin\alpha\sin\xi) & (\sin\alpha\cos\xi\sin\beta-\cos\alpha\cos\beta\ ) \end{bmatrix}$$

$$* \quad (60) \qquad \left\{S^r\right\} = \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\}$$

$$* \quad (61) \qquad \left\{R^r\right\}_e = -R^r \left\{A^{re}\right\}^T \left\{S^r\right\}$$

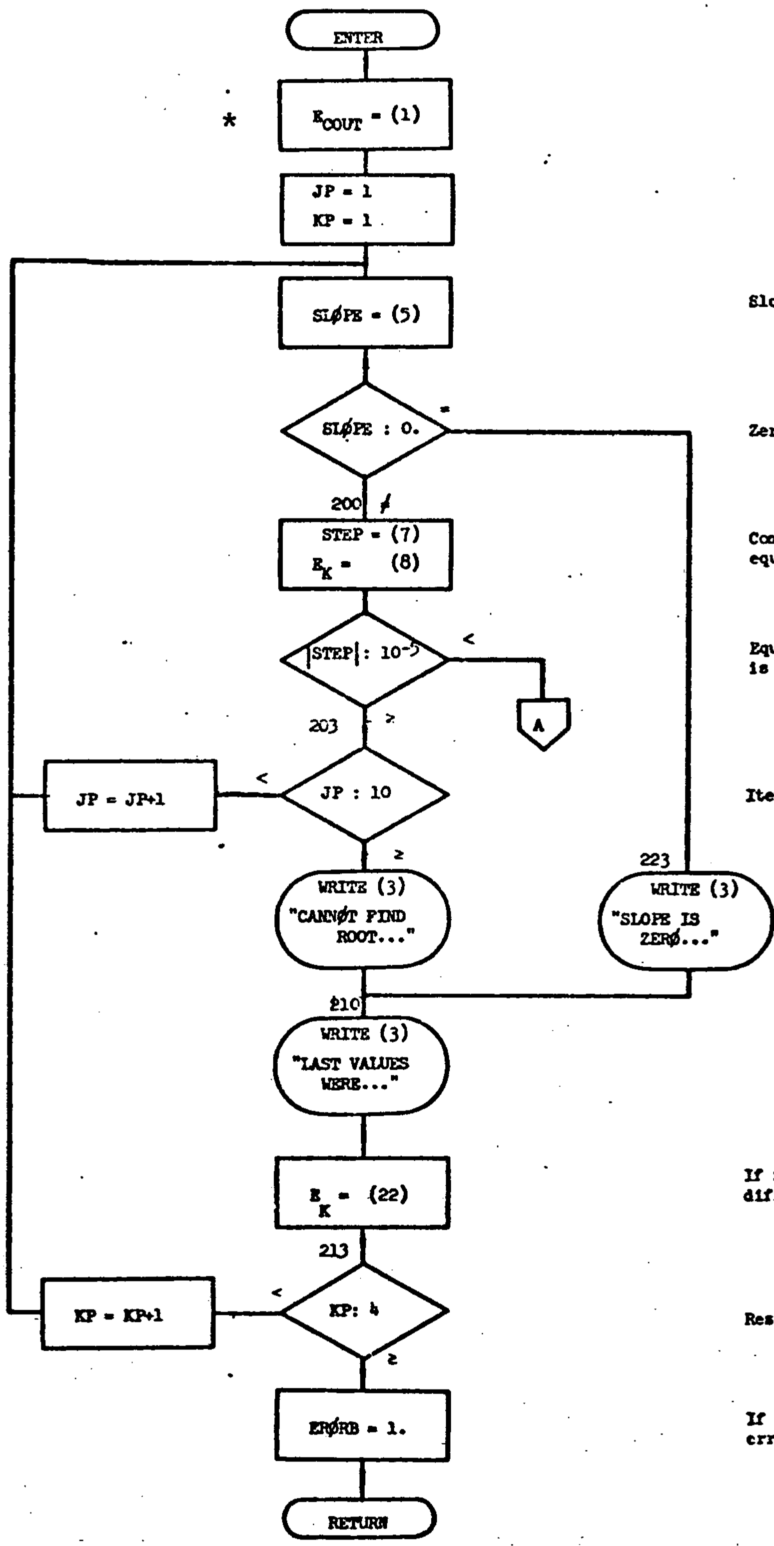
$$* \quad (62) \qquad \left\{a^r\right\}_e = \left\{R^r\right\}_e \Big/ R^r$$

$$* \quad (63) \qquad \left\{\dot{R}^r\right\}_e = -\left\{A^{re}\right\}^T \left(\dot{R}^r \left\{S^r\right\} + R^r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}\right)$$

$$(63.1) \qquad V^2 = \left\{\dot{R}^r\right\}_e^T \left\{\dot{R}^r\right\}_e$$

$$* \quad (64) \qquad \left\{b^r\right\}_e = \left\{\dot{R}^r\right\}_e / V$$

$$* \quad (65) \qquad \left\{\ddot{R}^r\right\}_e = -\left\{A^{re}\right\}^T \Big(\ddot{R}^r \left\{S^r\right\} + 2 \dot{R}^r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}$$
$$+ R^r \left\{\hat{\dot{\omega}}^r\right\}_r \left\{S^r\right\} + R^r \left\{\hat{\omega}^r\right\}_r \left\{\hat{\omega}^r\right\}_r \left\{S^r\right\}\Big)$$



Slope of Kepler's Equation.

Zero slope is error condition.

Compute one iteration of Kepler's equation using Newton's method.

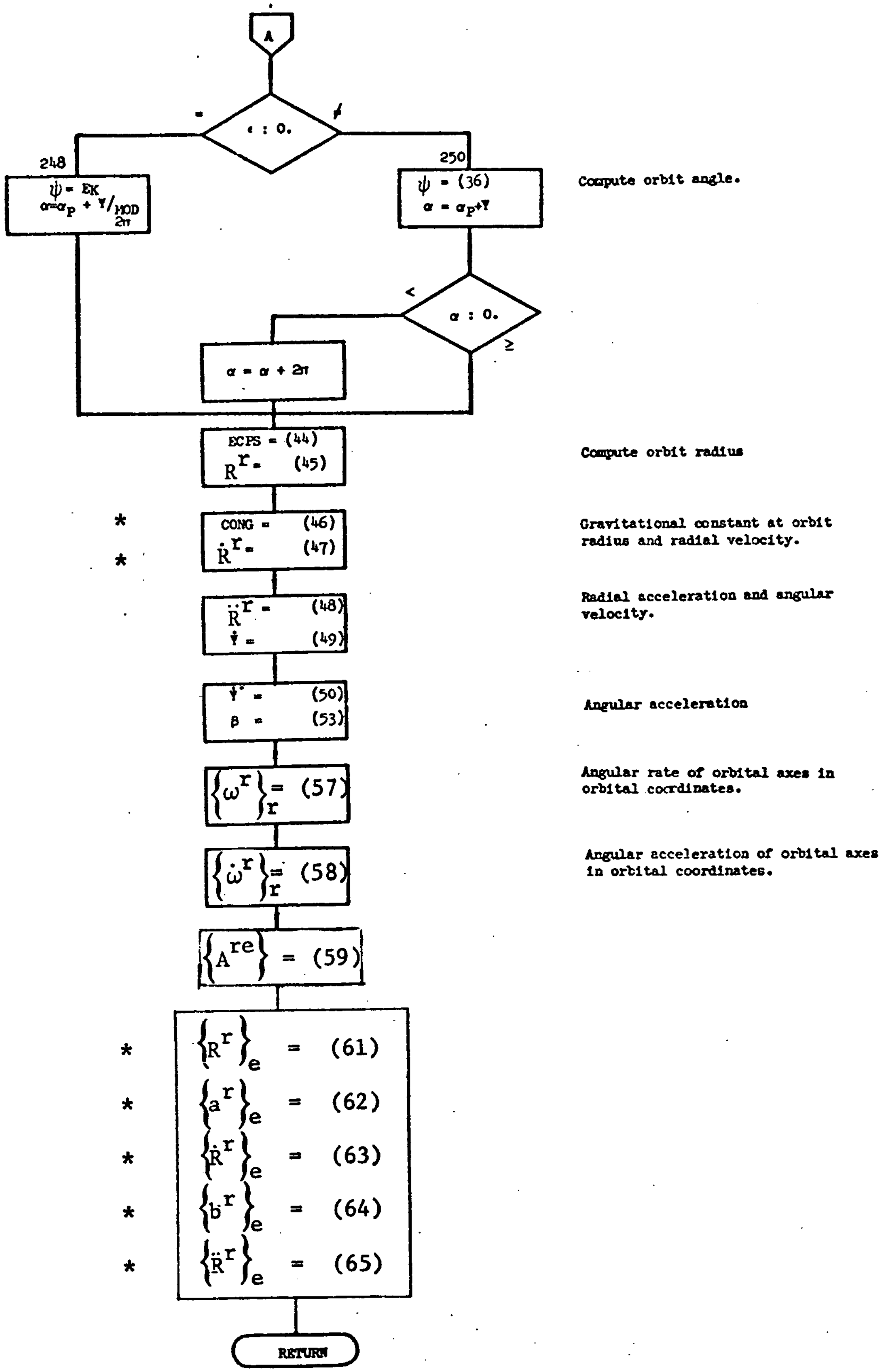
Equation is solved when iteration is less than 10^{-5}.

Iterate 10 times.

If no solution, restart with different initial value.

Restart 4 times maximum.

If no solution, exit with error flag set.


A
ε : 0.
=
≠
248
ψ = EK
α=αp + Y/MOD 2π
250
ψ = (36)
α = αp+Y
Compute orbit angle.
α : 0.
<
≥
α = α + 2π
ECPS = (44)
R r = (45)
Compute orbit radius
CONG = (46)
Ṙ r = (47)
Gravitational constant at orbit radius and radial velocity.
R̈ r = (48)
Ẏ = (49)
Radial acceleration and angular velocity.
Ÿ = (50)
β = (53)
Angular acceleration
{ω r}r = (57)
Angular rate of orbital axes in orbital coordinates.
{ω̇ r}r = (58)
Angular acceleration of orbital axes in orbital coordinates.
{A re} = (59)
* {R r}e = (61)
* {a r}e = (62)
* {Ṙ r}e = (63)
* {b r}e = (64)
* {R̈ r}e = (65)
RETURN

VIII. Control Subroutine

The UFSS Program contains a generalized control subroutine capability which allows the user to quickly and efficiently synthesize almost any conventional continuous attitude control system. The control system simulation capability is fully described in Reference [4]; this prsent section describes the interfaces between the dynamics and the controls subroutines.

Figure 8.1 presents a data flow diagram showing the various features of the Dynamics/Control Generalized Interface. The Sensor Interface generates the transformation matrix $A^{jcnj}_{\alpha\beta}$ and angular velocity components ω^{jcn}_{α} necessary to simulate the nth attitude and/or rate sensor mounted at node N (or at position $\bar{r}^{jN}$) on a given Body j of the spacecraft model; these quantities are then transmitted to the Control Routine for use in performing the attitude and rate error calculations.

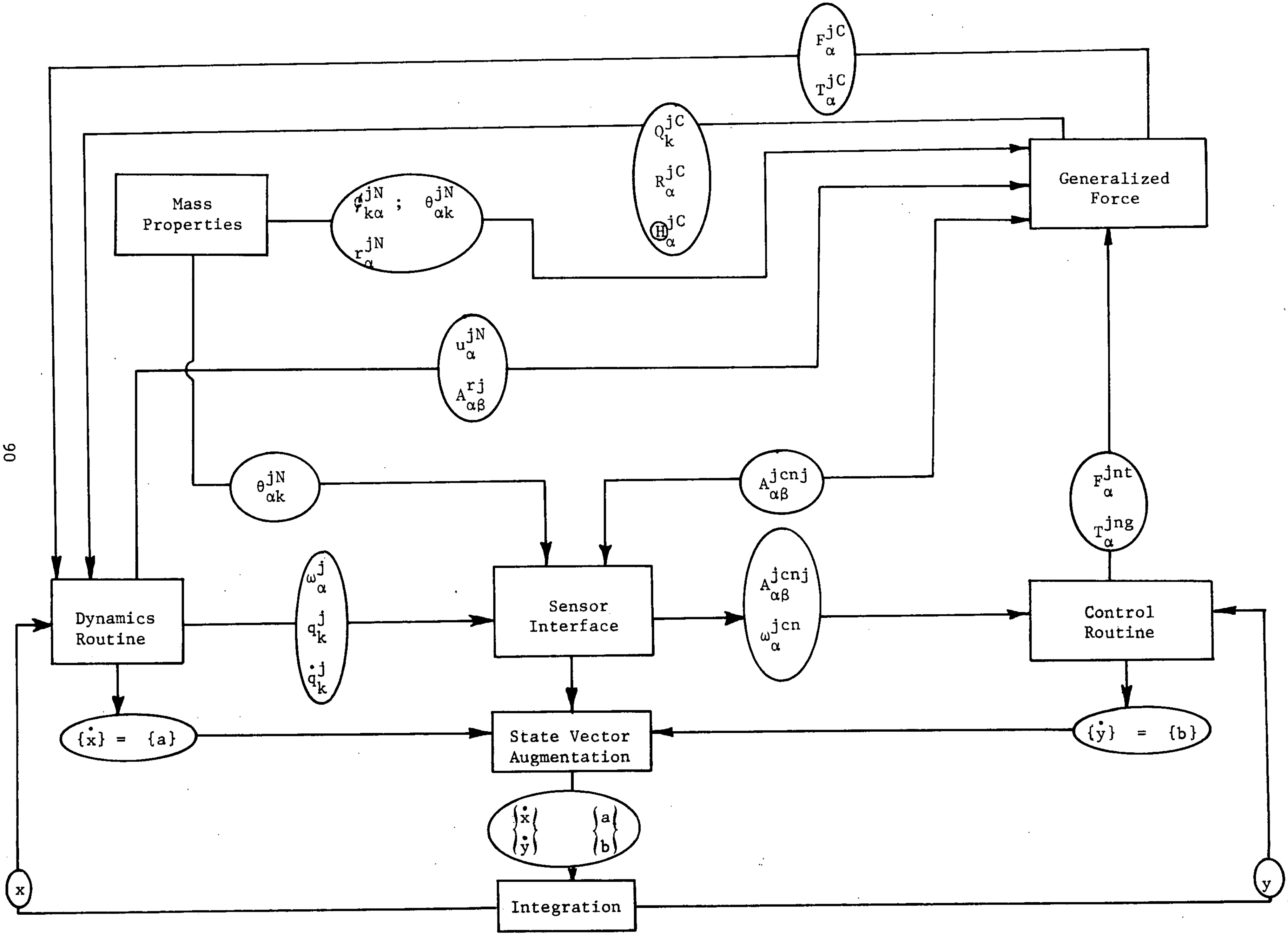
The Generalized Force Interface accepts as inputs the forces and torques produced by the Control Routine and transforms them into the generalized forces required by the Dynamics Routine.

If the control and steering laws (or any relationships existing in the control system) involve the solution of a differential equation, then the appropriate dependent variables are referred to as components of a control state vector. These control state vector equations must be added to the dynamic state vector equations and integrated simultaneously; this combining of the dynamic and control state vector equations is effected in the State Vector Augmentation Interface.

8.1 Sensor Interface

Assume that the nth control element (sensor or controller) for Body j is located at node N. In order to locate the elements sensitive axis, it is necessary to determine the instantaneous orientation of an axis frame located at node N and fixed in Body j. Specifically, if $\underline{e}^{jN}_{\alpha}$ is an axis frame based at node N and parallel to the $\underline{e}^{j}_{\alpha}$ frame for zero flexible motion of Body j, then it is necessary to determine the time-dependent transformation matrix $A^{jNj}_{\alpha\beta}$ relating $\underline{e}^{jN}_{\alpha}$ to $\underline{e}^{j}_{\beta}$ for Body j in its perturbed position:

Figure 8.1. Dynamics/Control Generalized Interface Data Flow Diagram



$$\underline{e}_\alpha^{jN} = A_{\alpha\beta}^{jNj} \; \underline{e}_\beta^{j}$$

where in general $A_{\alpha\beta}^{jNj}$ is a function of the q_ℓ^j $(\ell=1,2,\ldots,n_j)$.

When Body j undergoes flexural motion, the axis frame $\underline{e}_\alpha^{jN}$ rotates from the $\underline{e}_\alpha^{j}$ orientation to its final orientation as shown in Figure 8.2.

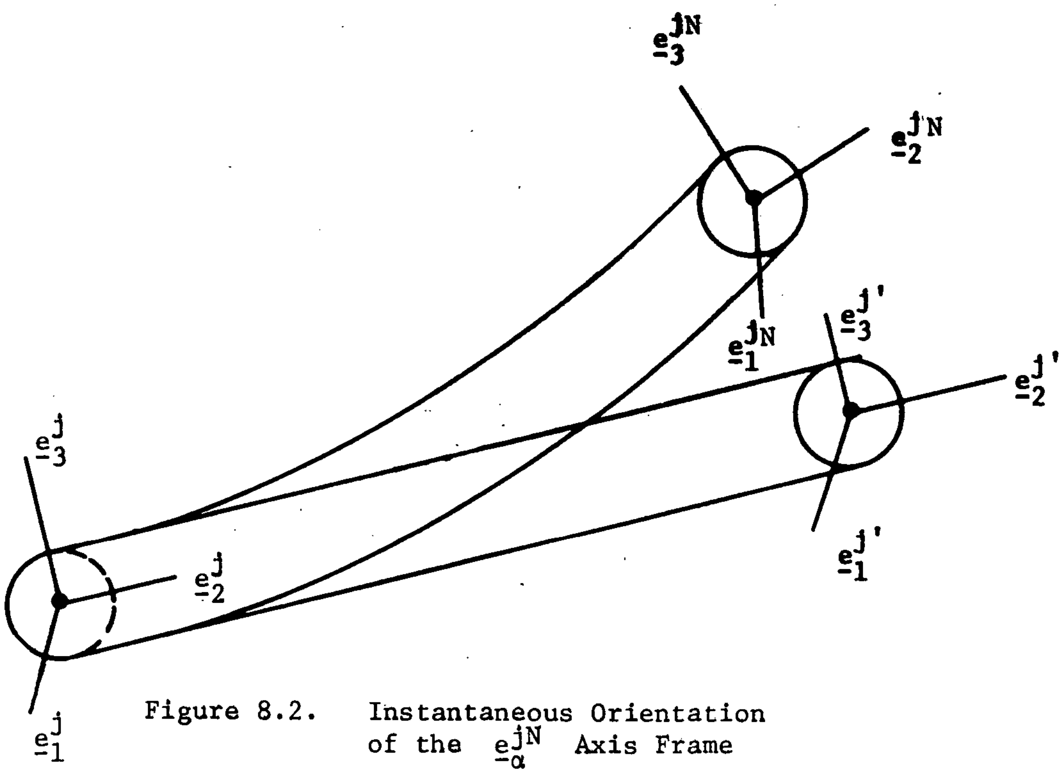


Figure 8.2. Instantaneous Orientation of the $\underline{e}_\alpha^{jN}$ Axis Frame

To a linear approximation, this rotation of the $\underline{e}_\alpha^{jN}$ frame is expressed by specifying three small rotations about an axis set $\underline{e}_\alpha^{j'}$ parallel to the $\underline{e}_\alpha^{j}$ set but located at node N . Calling these rotations ψ_α^{jN} , it follows that

$$\psi_\alpha^{jN} = \theta_{\alpha k}^{jN} \; q_k^j \tag{8-1}$$

where $\theta_{\alpha k}^{jN}$ is the $3 \times n_j$ matrix whose kth column contains the three modal rotation components at node N of Body j for its kth mode.

Since the ψ_{α}^{jN} are small rotations, it follows that

$$\underline{e}_{\alpha}^{jN} = A_{\alpha\beta}^{jNj} \; \underline{e}_{\beta}^{j}$$

where

$$A_{\alpha\beta}^{jNj} = \begin{bmatrix} 1 & \psi_3^{jN} & -\psi_2^{jN} \\ -\psi_3^{jN} & 1 & \psi_1^{jN} \\ \psi_2^{jN} & -\psi_1^{jN} & 1 \end{bmatrix}$$

or more simply:

$$A_{\alpha\beta}^{jNj} = \delta_{\alpha\beta} + \tilde{\psi}_{\alpha\beta}^{jN} \tag{8-2}$$

where $\delta_{\alpha\beta}$ is the unit tensor or identity matrix.

Now let us locate the nth control element reference frame, call it $\underline{e}_{\alpha}^{jcn}$, at node N. The frame $\underline{e}_{\alpha}^{jcn}$ is related to the body-fixed frame $\underline{e}_{\alpha}^{jN}$ by a constant input transformation $A_{\alpha\beta}^{jcnjN}$:

$$\underline{e}_{\alpha}^{jcn} = A_{\alpha\beta}^{jcnjN} \; \underline{e}_{\beta}^{jN}$$

where

$$A_{\alpha\beta}^{jcnjN} = (G_3^{jcn})_{\alpha\gamma} \, (G_2^{jcn})_{\gamma\sigma} \, (G_1^{jcn})_{\sigma\beta} \tag{8-3}$$

The $(G_{\rho}^{jcn})_{\alpha\beta}$ are formed from three user supplied Euler rotations θ_{ρ}^{jcn} as follows:

$$(G_1^{jcn})_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1^{jcn} & \sin\theta_1^{jcn} \\ 0 & -\sin\theta_1^{jcn} & \cos\theta_1^{jcn} \end{bmatrix} \tag{8-4}$$

$$(G_2^{jcn})_{\alpha\beta} \quad \begin{bmatrix} \cos\theta_2^{jcn} & 0 & -\sin\theta_2^{jcn} \\ 0 & 1 & 0 \\ \sin\theta_2^{jcn} & 0 & \cos\theta_2^{jcn} \end{bmatrix} \tag{8-5}$$

$$(G_3^{jcn})_{\alpha\beta} \quad \begin{bmatrix} \cos\theta_3^{jcn} & \sin\theta_3^{jcn} & 0 \\ -\sin\theta_3^{jcn} & \cos\theta_3^{jcn} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8-6}$$

Finally then, the nth control element reference frame $\underline{e}_\alpha^{jcn}$ is related to the Body j reference axis frame $\underline{e}_\beta^{j}$ through the time-dependent transformation matrix $A_{\alpha\beta}^{jcnj}$

$$\underline{e}_\alpha^{jcn} = A_{\alpha\beta}^{jcnj} \; \underline{e}_\beta^{j}$$

where

$$A_{\alpha\beta}^{jcnj} = A_{\alpha\gamma}^{jcnjN} \; A_{\gamma\beta}^{jNj} \quad . \tag{8-7}$$

It should be noted that if Body j is rigid, then $A_{\alpha\beta}^{jNj} = \delta_{\alpha\beta}$ so that $A_{\alpha\beta}^{jcnj} = A_{\alpha\beta}^{jcnjN}$, a constant matrix.

In addition to calculating the orientation matrix $A^{jcnj}_{\alpha\beta}$, the Sensor Interface calculates the inertial angular velocity $\bar{\omega}^{jcn} = \omega^{jcn} \underline{e}^{jcn}$ of the $\underline{e}^{jcn}_{\alpha}$ frame, where

$$\bar{\omega}^{jcn} = \bar{\omega}^{j} + \bar{\Omega}^{jNj}$$

with $\bar{\Omega}^{jNj}$ being the angular velocity of the $\underline{e}^{jcn}_{\alpha}$ frame relative to the $\underline{e}^{j}_{\beta}$ frame. In particular,

$$\bar{\Omega}^{jNj} = \dot{\psi}^{jN}_{\alpha} \underline{e}^{j}_{\alpha}$$

where

$$\dot{\psi}^{jN}_{\alpha} = \theta^{jN}_{\alpha k} \dot{q}^{j}_{k} \tag{8-8}$$

Thus,

$$\omega^{jcn}_{\alpha} = A^{jcnj}_{\alpha\beta}\{\omega^{j}_{\beta} + \dot{\psi}^{jN}_{\beta}\} \tag{8-9}$$

8.2 Generalized Force Interface

If the nth control element (located at node N) for Body j is a thruster, then the output of the Control Routine is the instantaneous value of the thrust vector components in the nth control element reference frame,

$$\bar{F}^{jnt} = F^{jnt}_{\alpha} \underline{e}^{jcn}_{\alpha} . \tag{8-10}$$

If the nth control element is a torquer (CMG), then the output of the Control Routine is the instantaneous value of the torque vector components in the nth control element reference frame,

$$\bar{T}^{jng} = T^{jng}_{\alpha} \underline{e}^{jcn}_{\alpha} \tag{8-11}$$

If the nth control element is a gimbal hinge torquer, the output of the Control Routine is the instantaneous value of the motor torque components about the hinge axes,

$$T\alpha^{jm}.$$

If Body j is a rigid body, then the output of the Generalized Force Interface is the total force and moment about the mass center

$$F_{\alpha}^{jC} = \sum_{n} A_{\alpha\beta}^{jcnj(T)} \; F_{\beta}^{jnt} \tag{8-12}$$

$$T_{\alpha}^{jC} = \sum_{n} \left\{ A_{\alpha\beta}^{jcnj(T)} \; T_{\beta}^{jng} + \tilde{r}_{\alpha\beta}^{jcn} \; A_{\beta\gamma}^{jcnj(T)} \; F_{\gamma}^{jnt} \right\}, \tag{8-13}$$

where $\bar{r}^{jcn} = r_{\alpha}^{jcn} \; \underline{e}_{\alpha}^{j}$ is the input position vector of the nth control element on Body j with respect to its mass center.

If Body j is a flexible body, then the Generalized Force Interface must calculate the required generalized forces. Let us first assume that the nth control element is a thruster at node N .

The virtual work done by the thruster force $d\bar{f}^{jtn}$ acting on the arbitrary mass element dm^{j} of Body j due to the virtual displacement

$$\delta\bar{R}^{i} + \delta\bar{u}^{j} + \delta\bar{\theta}^{j} \times (\bar{r}^{j} + \bar{u}^{j})$$

is

$$\delta W^{jnT} = \int_{B^{j}} d\bar{f}^{jtn} \cdot \left[\delta\bar{R}^{i} + \delta\bar{u}^{j} + \delta\bar{\theta}^{j} \times (\bar{r}^{j} + \bar{u}^{j})\right] \tag{8-14}$$

where

$$\begin{aligned} \delta\bar{R}^{i} &= \delta R_{\alpha}^{i} \; \underline{e}_{\alpha}^{r} \\ \delta\bar{u}^{j} &= \delta q_{\ell}^{j} \; \phi_{\ell\alpha}^{j} \; \underline{e}_{\alpha}^{j} \\ \delta\bar{\theta}^{j} &= \delta\theta_{\alpha}^{j} \; \underline{e}_{\alpha}^{jg} \end{aligned}$$

But, for a thruster mounted at node N ,

$$\bar{d}f^{jtn} = \bar{F}^{jtn} \; \delta \; (\bar{r}^{j} - \bar{r}^{jN}) \; d \; V^{j}$$

where

$\bar{F}^{jtn}$ = thrust vector from the nth thruster on Body j located at node N (in $\underline{e}_\alpha^{jcn}$ frame)

$\bar{r}^{jN}$ = position vector of node N relative to the Body j hinge (in $\underline{e}_\alpha^{j}$ frame)

dV^j = arbitrary volume element of Body j

$\bar{r}^j$ = position vector to an arbitrary point within dV^j relative to the Body j hinge (in $\underline{e}_\alpha^{j}$ frame)

$\delta(\bar{a} - \bar{a}_o)$ = Dirac delta function having the property that $\int_{B^j} f(\bar{a})\,\delta\,(\bar{a} - \bar{a}_o)\,dV^j = f(\bar{a}_o)$

Thus

$$\delta W^{jnT} = \bar{F}^{jtn} \cdot \left[\delta R_\alpha^i\, \underline{e}_\alpha^r + \delta q_\ell^j\, \phi_{\ell\alpha}^{jN}\, \underline{e}_\alpha^j + \delta\theta_\alpha^j\, \underline{e}_\alpha^{jg} \times (\bar{r}^{jN} + \bar{u}^{jN})\right]$$

or

$$\delta W^{jnT} = R_\alpha^{jnT}\, \delta R_\alpha^i + Q_k^{jnT}\, \delta q_k^j + Q(H)_\alpha^{jnT}\, \delta\theta_\alpha^j \tag{8-15}$$

so that

$$R_\alpha^{jnT} = \bar{F}^{jnt} \cdot \underline{e}_\alpha^r$$

$$Q_k^{jnT} = \bar{F}^{jnt} \cdot \underline{e}_\alpha^j\, \phi_{k\alpha}^{jN}$$

$$Q(H)_\alpha^{jnT} = \bar{F}^{jnt} \cdot \left[\underline{e}_\alpha^{jg} \times (\bar{r}^{jN} + \bar{u}^{jN})\right]$$

where R_α^{jnT}, Q_k^{jnT}, $Q(H)_\alpha^{jnT}$ are the thruster generalized forces associated with the generalized coordinates R_α^i, q_k^j and θ_α^j respectively.

Thus,

$$R^{jnT}_{\alpha} = F^{jnt}_{\beta} \quad \underline{e}^{jcn}_{\beta} \cdot \underline{e}^{r}_{\alpha}$$

or

$$R^{jnT}_{\alpha} = A^{rj}_{\alpha\gamma} \quad A^{jcnj(T)}_{\gamma\beta} \quad F^{jnt}_{\beta} \tag{8-16}$$

Likewise,

$$Q^{jnT}_{k} = F^{jnt}_{\beta} \quad \underline{e}^{jcn}_{\beta} \cdot \underline{e}^{j}_{\alpha} \quad \phi^{jN}_{k\alpha}$$

so that

$$Q^{jnT}_{k} = \phi^{jN}_{k\alpha} \quad A^{jcnj(T)}_{\alpha\beta} \quad F^{jnt}_{\beta} \tag{8-17}$$

Finally,

$$\begin{aligned} Q\text{Ⓗ}^{jnT}_{\alpha} &= F^{jnt}_{\beta} \quad \underline{e}^{jcn}_{\beta} \cdot \left\{ \underline{e}^{jg}_{\alpha} \times (r^{jN}_{\gamma} + u^{jN}_{\gamma}) \quad \underline{e}^{j}_{\gamma} \right\} \\ &= \underline{e}^{jg}_{\alpha} \cdot \left\{ (r^{jN}_{\gamma} + u^{jN}_{\gamma}) \quad \underline{e}^{j}_{\gamma} \times F^{jnt}_{\beta} \quad \underline{e}^{jcn}_{\beta} \right\} \\ &= G^{j(T)}_{\alpha\sigma} \quad \left\{ \tilde{r}^{jN}_{\sigma\gamma} + \tilde{u}^{jN}_{\sigma\gamma} \right\} \quad A^{jcnj(T)}_{\gamma\beta} \quad F^{jnt}_{\beta} \end{aligned}$$

But, transforming from hinge axis to Body j axes,

$$\text{Ⓗ}^{jnT}_{\alpha} = [G^{j(T)}]^{-1}_{\alpha\beta} \quad Q\text{Ⓗ}^{jnT}_{\beta}$$

so that

$$\text{Ⓗ}^{jnT}_{\alpha} = \left\{ \tilde{r}^{jN}_{\alpha\gamma} + \tilde{u}^{jN}_{\alpha\gamma} \right\} \quad A^{jcnj(T)}_{\gamma\beta} \quad F^{jnt}_{\beta} \tag{8-18}$$

Let us now assume that the nth control element is a torquer (CMG) at node N. In order to calculate the generalized forces due to an instantaneous torque $\bar{T}^{jGn}$ produced by this torque generator, one introduces two fictitious nodes N^+ and N^- located at the terminus of the vectors $1/2\ \bar{\varepsilon}$ and $-\ 1/2\ \bar{\varepsilon}$ respectively from node N. At node N^+ a force $\bar{F}^{jn+}$ is introduced and at node N^- a force $\bar{F}^{jn-}$ is introduced such that: (see Figure 8.3).

1) $\bar{F}^{jn-} = -\ \bar{F}^{jn+}$

2) $\bar{T}^{jGn} = 1/2\ \bar{\varepsilon} \times \bar{F}^{jn+} + (-\ 1/2\ \bar{\varepsilon}) \times \bar{F}^{jn-}$

$\qquad = 1/2\ \bar{\varepsilon} \times (\bar{F}^{jn+} - \bar{F}^{jn-})$

3) $|\ \bar{\varepsilon}\ | << 1$

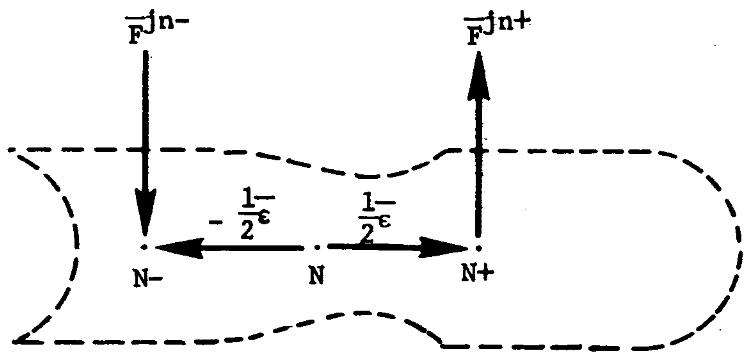


Figure 8.3. Introduction of Fictitious Nodes

In this case,

$$\delta W^{jnG} = (\bar{F}^{jN+} + \bar{F}^{jN-}) \cdot \delta R^i_\alpha\ \underline{e}^r_\alpha$$

$$+\ \bar{F}^{jN+} \cdot \delta q^j_\ell\ \phi^{jN+}_{\ell\alpha}\ \underline{e}^j_\alpha + \bar{F}^{jN-} \cdot \delta q^j_\ell\ \phi^{jN-}_{\ell\alpha}\ \underline{e}^j_\alpha$$

$$+\ \bar{F}^{jN+} \cdot \delta\theta^j_\alpha\ \underline{e}^{jg}_\alpha \times (\bar{r}^{jN+} + \bar{u}^{jN+})$$

$$+\ \bar{F}^{jN-} \cdot \delta\theta^j_\alpha\ \underline{e}^{jg}_\alpha \times (\bar{r}^{jN-} + \bar{u}^{jN-})$$

or

$$\delta W^{jnG} = R_\alpha^{jnG} \;\delta R_\alpha^{i} + Q_k^{jnG} \;\delta q_k^{j} + Q\,Ⓗ_\alpha^{jnG} \;\delta\theta_\alpha^{j} , \tag{8-19}$$

so that

$$R_\alpha^{jnG} = 0 \quad (\text{since} \;\; \bar{F}^{jN+} + \bar{F}^{jN-} \equiv 0) \tag{8-20}$$

$$Q_k^{jnG} = \bar{F}^{jN+} \cdot \phi_{k\alpha}^{jN+} \; \underline{e}_\alpha^{j} + \bar{F}^{jN-} \cdot \phi_{k\alpha}^{jN-} \; \underline{e}_\alpha^{j}$$

$$Q\,Ⓗ_\alpha^{jnG} = \bar{F}^{jN+} \cdot \underline{e}_\alpha^{jg} \times (\bar{r}^{jN+} + \bar{u}^{jN+}) + \bar{F}^{jN-} \cdot \underline{e}_\alpha^{jg} \times (\bar{r}^{jN-} + \bar{u}^{jN-})$$

Let us first examine Q_k^{jnG} ...

$$\bar{\phi}_k^{jN+} = \bar{\phi}_k^{jN} + \bar{\theta}_k^{jN} \times \tfrac{1}{2}\,\bar{\varepsilon}$$

$$\bar{\phi}_k^{jN-} = \bar{\phi}_k^{jN} + \bar{\theta}_k^{jN} \times (-\tfrac{1}{2}\,\bar{\varepsilon})$$

$$\begin{aligned} Q_k^{jnG} &= \bar{F}^{jN+} \cdot \left[\bar{\phi}_k^{jN} + (\bar{\theta}_k^{jN} \times \tfrac{1}{2}\,\bar{\varepsilon})\right] + \bar{F}^{jN-} \cdot \left[\bar{\phi}_k^{jN} + \bar{\theta}_k^{jN} \times (-\tfrac{1}{2}\,\bar{\varepsilon})\right] \\ &= \bar{\theta}_k^{jN} \cdot \left[\tfrac{1}{2}\,\bar{\varepsilon} \times (\bar{F}^{jN+} - \bar{F}^{jN-})\right] \\ &= \bar{\theta}_k^{jN} \cdot \bar{T}^{jng} \end{aligned}$$

Therefore

$$Q_k^{jnG} = \theta_{k\alpha}^{jN(T)} \; A_{\alpha\beta}^{jcnj(T)} \; T_\beta^{jnG} \tag{8-21}$$

Now let us examine $Q\,Ⓗ_\alpha^{jnG}$.

$$Q \textcircled{H}_{\alpha}^{jnG} = \bar{F}^{jN+} \cdot \underline{e}_{\alpha}^{jg} \times (\bar{r}^{jN} + \frac{1}{2} \bar{\varepsilon}) + \bar{F}^{jN-} \cdot \underline{e}_{\alpha}^{jg} \times (\bar{r}^{jN} - \frac{1}{2} \bar{\varepsilon})$$

$$+ \bar{F}^{jN+} \cdot \underline{e}_{\alpha}^{jg} \times (\bar{u}^{jN} + q_{\ell}^{j} \bar{\theta}_{\ell}^{jN} \times \frac{1}{2} \bar{\varepsilon})$$

$$+ \bar{F}^{jN-} \cdot \underline{e}_{\alpha}^{jg} \times \left\{ \bar{u}^{jN} + q_{\ell}^{j} \bar{\theta}_{\ell}^{jN} \times (- \frac{1}{2} \bar{\varepsilon}) \right\}$$

or

$$Q \textcircled{H}_{\alpha}^{jnG} = \underline{e}_{\alpha}^{jg} \cdot \left\{ \frac{1}{2} \bar{\varepsilon} \times (\bar{F}^{jN+} - \bar{F}^{jN-}) \right\}$$

$$+ \underline{e}_{\alpha}^{jg} \cdot (q_{\ell}^{j} \bar{\theta}_{\ell}^{jN} \times \frac{1}{2} \bar{\varepsilon}) \times \bar{F}^{jN+}$$

$$- \underline{e}_{\alpha}^{jg} \cdot (q_{\ell}^{j} \bar{\theta}_{\ell}^{jN} \times \frac{1}{2} \bar{\varepsilon}) \times \bar{F}^{jN-}$$

$$= \underline{e}_{\alpha}^{jg} \cdot \bar{T}^{jng} + \underline{e}_{\alpha}^{jg} \cdot (q_{\ell}^{j} \bar{\theta}_{\ell}^{jN} \times \bar{\varepsilon}) \times \bar{F}^{jN+}$$

Retaining only the lowest order term,

$$Q \textcircled{H}_{\alpha}^{jnG} = \underline{e}_{\alpha}^{jg} \cdot \bar{T}^{jng}$$

$$= G_{\alpha\beta}^{j(T)} A_{\beta\gamma}^{jcnj(T)} T_{\gamma}^{jng}$$

so that

$$\textcircled{H}_{\alpha}^{jnG} = A_{\alpha\beta}^{jcnj(T)} T_{\beta}^{jng} \tag{8-22}$$

Finally then, the total control generalized forces acting on the flexible Body j are:

$$R_{\alpha}^{jC} = \sum_{n} R_{\alpha}^{jnT} \tag{8-23}$$

$$Q_{k}^{jC} = \sum_{n} (Q_{k}^{jnT} + Q_{k}^{jnG}) \tag{8-24}$$

$$\textcircled{H}_{\alpha}^{jC} = \sum_{n} (\textcircled{H}_{\alpha}^{jnT} + \textcircled{H}_{\alpha}^{jnG}) \tag{8-25}$$

IX. Mass Properties Subroutine

The UFSS Program requires that the flexible deformation characteristics of the terminal bodies in the spacecraft model be expressed in terms of three-dimensional orthogonal functions. The standard source of these orthogonal functions is a Structural Dynamics Program (SDP) wherein each designated flexible body is modeled in the traditional structural dynamics sense as a lumped parameter system. The basic requirements placed on the structural dynamic model for a given flexible body are as follows:

1. The structural dynamics model for each simulated flexible body must basically consist of point masses only, with no significant lumped inertias admissible (no large lumped inertias are admissible because of the manner in which certain mass integrals over the flexible bodies are presently calculated). It is felt that this restriction is not a serious one particularly in view of the significant increase in calculations and cost necessary to allow for large lumped inertias in the structural dynamics model.
2. All three components of the lumped mass at a given node must be identical.
3. If sensors and/or thrusters are mounted on a simulated flexible body, then a node point (joint) must be positioned at each sensor and/or thruster location and modal rotations must be available at these node points.

The UFSS Program is designed so that choice of a structures program is completely arbitrary so long as a basic data set is obtainable. In particular, the SDP provides the following data set for each flexible body:

1. Model Description

   1.1 joint coordinates (r_α^j)

   1.2 joint mass values

2. Orthonormal Mode Specification

2.1 eigenvalues associated with each mode

2.2 joint deflections and rotations for each mode ($\theta^j_{k\alpha}$ and $\theta^j_{\alpha k}$)

It is important to note that no development work was intended on the SDP. Depending on the particular choice of a structures program, it is only necessary to appropriately reformat the output to agree with the format of the Structures Tape as given in the UFSSP Users Manual.

The Mass Properties Routine utilizes the modal and structural dynamics model data described above to produce the special mass property quantities required by the UFSSP for each flexible body of the spacecraft model. In particular, the following quantities are calculated:

$$m^j = \int_{B^j} dm^j = \text{Mass of Body } j \tag{9-1}$$

$$\bar{d}^j = \frac{1}{m^j} \int_{B^j} \bar{r}^j \, dm^j = \sum_{\beta=1}^{3} d^j_\beta \, \underline{e}^j_\beta = - \bar{\ell}^{ji} \tag{9-2}$$

$$\bar{\Phi}^j_\ell = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_\ell \, dm^j = \sum_{\beta=1}^{3} \Phi^j_{\ell\beta} \, \underline{e}^j_\beta \quad (\ell=1,2,3\ldots n_j) \tag{9-3}$$

$$\bar{Y}^j_\ell = \frac{1}{m^j} \int_{B^j} \bar{r}^j \times \bar{\phi}^j_\ell \, dm^j = \sum_{\beta=1}^{3} Y^j_{\ell\beta} \, \underline{e}^j_\beta \quad (\ell=1,2,3\ldots n_j) \tag{9-4}$$

$$\bar{Z}^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \times \bar{\phi}^j_\ell \, dm^j = \sum_{\beta=1}^{3} Z^j_{k\ell\beta} \, \underline{e}^j_\beta \quad (k,\ell=1,2,3\ldots n_j) \tag{9-5}$$

$$M^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \cdot \bar{\phi}^j_\ell \, dm^j \quad (k,\ell=1,2,3\ldots n_j) = \delta_{k\ell}/m^j \tag{9-6}$$

$$\bar{\bar{B}}^j_\ell = \frac{1}{m^j} \int_{B^j} \bar{r}^j \, \bar{\phi}^j_\ell \, dm^j = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} B^j_{\ell\alpha\beta} \, \underline{e}^j_\alpha \, \underline{e}^j_\beta \tag{9-7}$$

$$(\ell=1,2,3,\ldots n_j)$$

$$\bar{\bar{C}}^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \; \bar{\phi}^j_\ell \; dm^j = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} C^j_{k\ell\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta \tag{9-8}$$

$$(k, \ell = 1, 2, 3, \ldots n_j)$$

$$D^j_\ell = \frac{1}{m^j} \int_{B^j} \bar{r}^j \cdot \bar{\phi}^j_\ell \; dm^j \quad (\ell = 1, 2, 3, \ldots n_j) \tag{9-9}$$

$$\bar{\bar{N}}^j_\ell = \frac{1}{m^j} \int_{B^j} \bar{\bar{\delta}} \; (\bar{r}^j \cdot \bar{\phi}^j_\ell) - \bar{r}^j \; \bar{\phi}^j_\ell \; dm^j$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} N^j_{\ell\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta \; (\ell = 1, 2, 3, \ldots n_j) \tag{9-10}$$

where $N^j_{\ell\alpha\beta} = D^j_\ell \; \delta_{\alpha\beta} - B^j_{\ell\alpha\beta}$

$$\bar{\bar{E}}^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\bar{\delta}} \; (\bar{\phi}^j_k \cdot \bar{\phi}^j_\ell) - \bar{\phi}^j_k \; \bar{\phi}^j_\ell \; dm^j$$

$$= \sum_{\alpha=1}^{3} \sum_{\alpha=1}^{3} E^j_{k\ell\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta \quad (k, \ell = 1, 2, 3, \ldots n_j) \tag{9-11}$$

where $E^j_{k\ell\alpha\beta} = M^j_{k\ell} \delta_{\alpha\beta} - C^j_{k\ell\alpha\beta}$

$$\bar{\bar{I}}^{jf} = \frac{1}{m^j} \int_{B^j} \bar{\bar{\delta}} \; (\bar{r}^j \cdot \bar{r}^j) - \bar{r}^j \; \bar{r}^j \; dm^j$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} I^{jf}_{\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta = \frac{1}{m^j} \; \bar{\bar{I}}^j \tag{9-12}$$

where $\bar{\bar{I}}^j$ is the standard inertia tensor for Body j with respect to the $\underline{e}_\beta^j$ frame with origin at the Body j hinge.

In several of the above equations,

$$\bar{\bar{\delta}} = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \delta_{\alpha\beta} \; \underline{e}_\alpha^j \; \underline{e}_\beta^j = \text{Kronëcker delta tensor} \tag{9-13}$$

$$\delta_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Finally, the generalized stiffness matrix for Body j is computed as follows:

$$K_{k\ell}^j = \frac{1}{m^j} \; \delta_{k\ell} \; {\omega_{(\ell)}^j}^2 \tag{9-14}$$

so that $K_{k\ell}^j$ is a diagonal matrix whose ℓth diagonal element is the square of the ℓth eigenvalue for Body j divided by the mass of Body j.

The integrations indicated in Equations (9-1) through (9-12) are replaced by a summation over the nodes (joints) of each flexible body when the indicated quantities are calculated.

X. Definition of Quantities

The following tables define the quantities used in the UFSS Program. "Body (F,R)" specifies if the quantity is used for a flexible or a rigid body; "Comp" specifies the program area in which the quantity is computed and "Used" specifies those program areas where the quantity is used — according to the following code:

1. Orbit Subroutine
2. Disturbance Subroutine
3. Control Subroutine
4. Auxiliary Calculations
5. Sequencing Algorithm
6. Rigid Combining Algorithm
7. Flexible Combining Algorithm
8. State Vector
9. Mass Properties Subroutine
10. Input
11. Initial Conditions

The units are mass (M), length (L), time (T) and the derived unit force (F) equal to MLT^{-2}.

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	time	Comp	Used	Units
A^j	–	✓	✓	Area of flat plate	N	10	2	L^2
$A^{1r}_{\alpha\beta}$	3, 3		✓	Transformation Matrix (T.M.), Body 1 axes to reference axes	Y	11, 8	2, 3, 4, 6, 8	–
$\dot{A}^{1r}_{\alpha\beta}$	3, 3		✓	Time derivative of $A^{ir}_{\alpha\beta}$	Y	4	8	T^{-1}
$A^{ji}_{\alpha\beta}$	3, 3	✓	✓	T.M., Body j axes to Body i axes	Y	4	6, 7	–
$A^{jjc}_{\alpha\beta}$	3, 3	✓	✓	T.M., Body j axes to nominal Body j axes	Y	4	6, 7	–
$A^{jr}_{\alpha\beta}$	3, 3	✓	✓	T.M., Body j axes to reference axes	Y	4	2, 3, 4, 6, 7, 8	–
$A^{je}_{\alpha\beta}$	3, 3	✓	✓	T.M., Body j axes to inertial axes	Y	4	2, 3	–
$A^{re}_{\alpha\beta}$	3, 3			T.M., Reference axes to inertial axes	Y	1	2, 3	–
$A^{jcnj}_{\alpha\beta}$	3, 3	✓	✓	T.M., n^{th} controller axes to Body j axes	Y	3	3	–
$A^{jcnjN}_{\alpha\beta}$	3, 3 --	✓	✓	T.M., n^{th} controller nominal axes to Body j axes	N	10	3	–
$A^{jNj}_{\alpha\beta}$	3, 3	✓		T.M., n^{th} controller axes to its nominal axes	Y	3	3	–
a^r_α	3	·	·	unit vector parallel to R^r_α	Y	1	2, 3	–
$B^j_{k\alpha\alpha}$	n_j, 3, 3	✓		Special Mass property	N	9	2	L^2
b^r_α	3	·	·	unit vector parallel to $\dot{R}^r_\alpha$	Y	1	2	–
$b^j_{k\alpha}$	n_j, 3	✓		computed matrix	Y	2	2	L^2
B^j_α	3	✓		Components of normal to flat plate	N	10	2	–
$B^{A11} \rightarrow B^{A33}$	...	✓	✓	Sub-matrices of System A	Y	4	6, 7	...
$B^{B11} \rightarrow B^{B33}$	...	✓	✓	Sub-matrices of System B	Y	4	6, 7	...
$B^{C11} \rightarrow B^{C33}$	...	✓	✓	Sub-matrices of System C	Y	6, 7	6, 7	...
$C^{A1} \rightarrow C^{A3}$	...	✓	✓	Sub-matrix of System A	Y	4	6, 7	...
$C^{B1} \rightarrow C^{B3}$	...	✓	✓	Sub-matrix of System B	Y	4	6, 7	...

SYMBOL	Subscript Range	Body F	Body R	DESCRIP ION	time	Comp	Used	Units
$C^{C1} \rightarrow C^{C3}$	. . .	✓	✓	Sub-matrix of System C	Y	6,7	6, 7	. . .
$C_\alpha^{jf1} \rightarrow C_\alpha^{jf3}$	3		✓	Coefficients for Prescribed Force	N	10	2	. . .
$C_\alpha^{jt1} \rightarrow C_\alpha^{jt3}$	3		✓	Coefficients for Prescribed Torque	N	10	2	. . .
$C_{\alpha\beta}^{jn}$	3, 3	✓	✓	Matrix of gimbal damping constants	N	10	4	$FL/(Rad/T)^n$
A_{kk}^{j}	n_j, n_j	✓		computed matrix	Y	2	2	L^2
$C_{k\ell\alpha\beta}^{j}$	$n_j, n_j, 3, 3$	✓		Special mass property	N	9	2	L^2
D^{β}	-	.	.	Mean no. of days after autumnal equinox	N	10	1, 2	(Days)
D_k^j	n_j	✓		Special mass property	N	9	2	L^2
d_α^j	3	✓		Special mass property $(= -\ell_\alpha^{ji})$	N	9	2, 4, 7	L
$E_{k\ell\alpha\beta}^{j}$	$n_j, n_j, 3, 3$	✓		Special mass property	N	9	4	L^2
$\underline{e}_\alpha^{c}$	3	.	.	Unit vectors (u.v.) for inertial frame				
$\underline{e}^{j12}, \underline{e}^{j13}, \underline{e}^{j23}$	-		✓	u.v. normal to sides of parallelepiped				
$\underline{e}_\alpha^{j}$	3	✓	✓	u.v. for axes fixed in Body j				
$\underline{e}_\alpha^{jc}$	3	✓	✓	u.v. for Body j nominal axes				
$\underline{e}_\alpha^{jf}$	3	✓	✓	u.v. defining flow for disturbances				
$\underline{e}_\alpha^{r}$	3	.	.	u.v. for orbital reference frame				
$\underline{e}_\alpha^{jcn}$	3	✓	✓	u.v. for n^{th} controller frame on Body j				
$\underline{e}_\alpha^{jg}$	3	✓	✓	u.v. for Body j gimbal axes				
$\underline{e}_\alpha^{jN}$	3	✓	✓	u.v. for n^{th} controller nominal frame on Body j				
F_α^{je}	3		✓	Total external force on Body j	Y	4	6	F
F_α^{jnt}	3	✓	✓	Force vector for n^{th} thruster on Body j	Y	3	3	F

SYMBOL	Subscript Range	Body F	Body R	DESCRIP ION	time	Comp	Used	Units
F_α^{jD}	3		✓	Total disturbance force on Body j	Y	2	4	F
F_α^{jG}	3		✓	Gravity force on Body j	Y	2	2	F
F_α^{jA}	3		✓	Aerodynamic force on Body j	Y	2	2	F
F_α^{jS}	3		✓	Solar force on Body j	Y	2	2	F
F_α^{jP}	3		✓	Prescribed force on Body j	Y	2	2	F
F_α^{jC}	3		✓	Control force on Body j	Y	3	4	F
$G_{\alpha\beta}^{j}$	3, 3	✓	✓	T.M. for Body j	Y	4	6, 7	—
$G_{\alpha\beta}^{j+}$	3, P_j	✓	✓	Sub-matrix of $G_{\alpha\beta}^{j}$	Y	4	6, 7	—
$\dot{G}_{\alpha\beta}^{j}$	3, 3	✓	✓	Time derivative of $G_{\alpha\beta}^{j}$	Y	4	6, 7	T^{-1}
$[G^{j(T)}]_{\alpha\beta}^{-1}$	3, 3	✓	✓	Inverse of the transpose of $G_{\alpha\beta}^{j}$	Y	4	4, 7	—
$G_{\alpha\beta}^{j1} \rightarrow G_{\alpha\beta}^{j3}$	3, 3	✓	✓	T.M. for Body j	Y	4	4	—
$G_{\alpha\beta}^{jc1} \rightarrow G_{\alpha\beta}^{jc3}$	3, 3	✓	✓	T.M. for Body j	Y	4	4	—
$(G_1^{jcn})_{\alpha\beta} \rightarrow (G_3^{jcn})_{\alpha\beta}$	3, 3	✓	✓	T.M. for n^{th} controller on Body j	N	3	3	—
$G_{\alpha\beta}^{jc}$	3, 3	✓	✓	T.M. for Body j	Y	4	6, 7	—
$\dot{G}_{\alpha\beta}^{jc}$	3, 3	✓	✓	Time derivative of $G_{\alpha\beta}^{jc}$	Y	4	6, 7	—
h^{jc}			✓	Height of cylinder shape	N	10	2	L
$H^{j1} \rightarrow H^{j12}$	. . .	✓		Auxiliary flexible quantities	Y	4	2, 4, 7	. . .
$I_{\alpha\beta}^{j}$	3, 3		✓	Body j centroidal inertia matrix	N	10	2, 6	ML^2

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	time	Comp	Used	Units
$I^{jf}_{\alpha\beta}$	3, 3	✓		Body j inertia matrix about its hinge	N	9	2, 4	L^2
J_2	–	.	.	Second harmonic of orbited body's grav. pot.	N	10	1	–
$K^j_{k\ell}$	n_j, n_j	✓		Generalized stiffness matrix for Body j	N	9	4	L^2T^{-2}
$K^{jn}_{\alpha\beta}$	3, 3	✓	✓	Matrix of gimbal spring constants	N	10	4	$FL/(Rad)^n$
L^{jf}_{α}	3		✓	Position vector of geometric center from mass center for disturbance shapes	N	10	2	L
ℓ^{ij}_{α}	3	✓	✓	Vector from Body i mass center to Body j hinge	Y	10	4, 6, 7	L
$L^{j1} \rightarrow L^{j3}$	–		✓	Lengths of parallelepiped sides	N	10	2	L
$\dot{\ell}^{ij}_{\alpha}$	3	✓	✓	first time derivative of ℓ^{ij}_{α}	Y	10	4, 6, 7	LT^{-1}
$\ddot{\ell}^{ij}_{\alpha}$	3	✓	✓	second " " " "	Y	10	4, 6, 7	LT^{-2}
M^j_{α}	3		✓	Magnetic dipole for Body j	N	10	2	ML^2T^{-2}/oersted
m^j	–	✓	✓	Mass of Body j	N	10	1, 2, 4, 9	M
M^e	–	.	.	Magnitude of Earth's Magnetic field	N	10	2	$L^3 \cdot$ oersted
$M^j_{k\ell}$	n_j, n_j	✓		Generalized mass matrix for Body j	N	9	7	L^2
N	–	.	.	Total number of bodies in the model	N	10	5	–
$N^j_{\ell\alpha\beta}$	n_j, 3, 3	✓		Special mass property	N	9	4	L^2
n_j	–	✓		Number of flexible d.o.f. for Body j	N	10	4, 5, 7, 9	–
P $P^1 \rightarrow P^{21}, P^{100}$	. . .		✓	Auxiliary quantities	Y	6	6	. . .
P^s	–	✓	✓	Solar pressure constant	N	10	2	FL^{-2}
ρ_j	$(0 \leq \rho_j \leq 3)$	✓	✓	Number of rotational d.o.f. for Body j	N	10	4, 5	–
q^j_k	n_j	✓		Generalized modal coordinate for Body j	Y	11, 8	4, 7	–

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	Time	Comp	Used	Units
$\dot{q}_k^j$	n_j	✓		first time derivative of q_k^j	Y	11, 8	4, 7	T^{-1}
$\ddot{q}_k^j$	n_j	✓		second time derivative of q_k^j	Y	8	8	T^{-2}
$Q^1 \rightarrow Q^5$	. . .	✓		Computed matrices	Y	7	7	. . .
Q_k^{je}	n_j	✓		Total external generalized force for q_k^j	Y	4	7	ML^2T^{-2}
Q_k^{jD}	n_j	✓		Total disturbance " " " "	Y	2	4	ML^2T^{-2}
Q_k^{jA}	n_j	✓		Aerodynamic " " " "	Y	2	2	ML^2T^{-2}
Q_k^{jC}	n_j	✓		Control " " " "	Y	3	4	ML^2T^{-2}
Q_k^{jS}	n_j	✓		Solar " " " "	Y	2	2	ML^2T^{-2}
Q_k^{jG}	n_j	✓		Gravity " " " "	Y	2	2	ML^2T^{-2}
R_b	—	.	.	Radius of the orbited body	N	10	1	L
R_α^1	3		✓	Position vector of Body 1 mass ctr.	Y	11, 8	2, 4	L
$\dot{R}_\alpha^1$	3		✓	first derivative of R_α^1	Y	11, 8	4	LT^{-1}
$\ddot{R}_\alpha^1$	3		✓	second " " "	Y	8	8	LT^{-2}
R_α^j	3		✓	Position vector of Body j mass ctr.	Y	4	2, 6, 7	L
$\dot{R}_\alpha^j$	3		✓	first derivative of R_α^j	Y	4	6, 7	LT^{-1}
R_α^r	3	.	.	Position vector of the reference frame	Y	1	1, 2	L
R^{js}, R^{jc}	—		✓	Radius of sphere, cylinder shapes	N	10	2	L
R_α^{je}	3	✓		Total external generalized force for R_α^j	Y	4	7	MLT^{-2}
R_α^{jA}	3	✓		Aerodynamic " " " "	Y	2	2	MLT^{-2}

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	time	Comp	Used	Units
R^{jD}_{α}	3	✓		Total disturbance generalized force for R^{j}_{α}	Y	2	4	$ML\ T^{-2}$
R^{jG}_{α}	3	✓		Gravity " " " "	Y	2	2	MLT^{-2}
R^{jS}_{α}	3	✓		Solar " " " "	Y	2	2	MLT^{-2}
R^{j}_{α}	3	✓		Aerodynamic " " " "	Y	2	2	MLT^{-2}
r^{j}_{α}	3	✓		Vector to a mass pt. in Body j	N	.	.	L
r^{jcn}_{α}	3	✓	✓	vector to the n^{th} controller on Body j	N	10	3	L
$S^{ij1} \rightarrow S^{ij15}$	. . .	✓		Auxiliary quantities	Y	4	7	. . .
t^{G}	—	.	.	Mean time in sec. after noon (GMT)	N	10	2	(sec)
t_{n}	—	.	.	time of first nodal crossing	N	10	1	T
T^{ji}_{α}	3	✓	✓	Torque on Body j due to Body i	Y	4	6, 7	FL
T^{jnq}_{α}	3	✓	✓	Torque from the n^{th} torquer on Body j	Y	3	3	FL
T^{jh}_{α}	3	✓	✓	Gimbal hinge torque for Body j	Y	4	4	FL
T^{js}_{α}	3	✓	✓	Gimbal spring " " "	Y	4	4	FL
T^{jd}_{α}	3	✓	✓	Gimbal damper " " "	Y	4	4	FL
T^{jm}_{α}	3	✓	✓	Gimbal motor " " "	Y	4	4	FL
T^{je}_{α}	3		✓	Total external torque on Body j	Y	4	6	FL
T^{jD}_{α}	3		✓	" disturbance " " "	Y	2	4	FL
T^{jA}_{α}	3		✓	Aerodynamic " " "	Y	2	2	FL
T^{jC}_{α}	3		✓	Control " " "	Y	3	4	FL
T^{jG}_{α}	3		✓	Gravity " " "	Y	2	2	FL

SYMBOL	Subscript Range	Body F	Body R	DESCRIF . ION	time	Comp	Used	Units
T_α^{jM}	3		✓	Magnetic torque on Body j	Y	2	2	F L
T_α^{jP}	3		✓	Prescribed " " "	Y	2	2	F L
T_α^{jS}	3		✓	Solar " " "	Y	2	2	F L
V_{ke}^{j}	n_j, n_j	✓		Generalized Damping matrix for Body j	N	10	7	$L^2 T^{-1}$
u_α^{j}	3	✓		Deformation of a point in Body j	Y		(output)	L
u_α^{jN}	3	✓		Deformation of N^{th} joint in Body j	Y		(output)	L
$Y_{k\beta}^{j}$	n_j, 3	✓		Special mass property	N	9	4	L^2
$Z_{k\ell\alpha}^{j}$	n_j, n_j, 3	✓		Special mass property	N	9	4	L^2
α_0	–	.	.	Orbit angle from line of nodes at time zero	N	10	1	(Rad)
α_p	–	.	.	Orbit angle between ascending node and perigee	N	10	1	(Rad)
γ		.	.	Orbited body's gravitational constant	N	10	1	$L^3 T^{-2}$
$\delta_{\alpha\beta}$	3, 3	.	.	Kronecker delta	N			
ε	–	.	.	Orbital eccentricity	N	10	1	–
η_α^{j}	3	✓		Position vector of Body j hinge	Y	2	2	L
θ_γ^{j}	ρ_j	✓	✓	Gimbal angles for Body j	Y	11, 8	4	(Rad)
$\dot{\theta}_\gamma^{j}$	ρ_j	✓	✓	First derivative of θ_γ^{j}	Y	11, 8	4	(Rad)/T
$\ddot{\theta}_\gamma^{j}$	ρ_j	✓	✓	Second " " "	Y	8	8	(Rad)/T^2
θ_α^{jen}	3	✓	✓	Euler angles to locate the n^{th} controller nominal frame	N	10	3	(Rad)
θ_α^{jo}	3	✓	✓	Euler angles to locate the Body j nominal frame	Y	10	4	(Rad)

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	time	Comp	Used	Units
$\dot{\theta}_{\alpha}^{jc}$	3	✓	✓	First derivative of θ_{α}^{jc}	Y	10	4	(Rad)/T
$\ddot{\theta}_{\alpha}^{jc}$	3	✓	✓	Second " " "	Y	10	4	(Rad)/T^2
$\ddot{\theta}_{\ell}^{j\hat{s}}$	. . .	.	.	Degrees of freedom of the Body j branch system (System A)	Y	6,7	6,7	. . .
$\ddot{\theta}_{n}^{i\hat{\ell}}$	. . .	.	.	Degrees of freedom of the Body i limb system (System B)	Y	6,7	6,7	. . .
$\ddot{\theta}^{i\hat{f}}$	. . .	.	.	Degrees of freedom of the combined system (System C)	Y	6,7	6,7	. . .
$\theta_{\alpha k}^{jN}$	3, n_j	✓		Modal rotations at joint N of Body j in mode k	Y	(SDP)	3	(Rad)
$Ⓗ_{\alpha}^{je}$	3	✓		Total external generalized force for θ_{α}^{j}	Y	4	7	ML^2T^{-2}
$Ⓗ_{\alpha}^{jD}$	3	✓		" disturbance " " " "	Y	2	4	ML^2T^{-2}
$Ⓗ_{\alpha}^{jG}$	3	✓		Gravity " " " "	Y	2	2	ML^2T^{-2}
$Ⓗ_{\alpha}^{jA}$	3	✓		Aerodynamic " " " "	Y	2	2	ML^2T^{-2}
$Ⓗ_{\alpha}^{j}$	3	✓		Control " " " "	Y	3	4	ML^2T^{-2}
$Ⓗ_{\alpha}^{jS}$	3	✓		Solar " " " "	Y	2	2	ML^2T^{-2}
$\mathfrak{I}$	—	.	.	Orbit inclination	N	10	1	(Rad)
ρ	—	.	.	Atmospheric density	N	10	2	ML^{-3}
σ^{j}	—	.	.	Aerodynamic reflection coefficient	N	10	2	—
σ^{js}	—	.	.	" " "	N	10	2	—
σ^{je}	—	.	.	" " "	N	10	2	—
$\sigma^{j12}, \sigma^{j13}, \sigma^{j23}$	—	.	.	" " "	N	10	2	—
ν^{j}				Solar reflectivity coefficient	N	10	2	—
ν^{js}				" " "	N	10	2	—
ν^{je}				" " "	N	10	2	—
$\nu^{j12}, \nu^{j13}, \nu^{j23}$				" " "	N	10	2	—

SYMBOL	Subscript Range	Body F	Body R	DESCRIPTION	time	Comp	Used	Units
$\phi^{j}_{k\alpha}(r^{j}_{\beta})$	n_j, 3	✓		Modal deflections of point on Body j located at position r^{j}_{β} in mode k	N	(SDP)	4	L
$\phi^{jN}_{k\alpha}$	n_j, 3	✓		Modal deflections at joint N of Body j in mode k	N	(SDP)	4, 3	L
$\Phi^{j}_{k\alpha}$	n_j, 3	✓		Special mass property	N	9	4, 7	L
ω^{j}_{α}	3	✓	✓	Inertial angular velocity of Body j	Y	4	4, 6, 7	(Rad)/T
$\dot{\omega}^{j}_{\alpha}$	3	✓	✓	First derivative of ω^{j}_{α}	Y	.	.	(Rad)/T^2
ω^{jcn}_{α}	3	✓	✓	Inertial angular velocity of n^{th} controller frame on Body j	Y	3	3	(Rad)/T
ω^{jf}	–		✓	Circular frequency for prescribed force	N	10	2	(Rad)/T
ω^{jt}	–		✓	" " " " torque	N	10	2	(Rad)/T
ω^{1}_{α}	3		✓	Inertial angular velocity of Body 1	Y	11, 8	4	(Rad)/T
$\dot{\omega}^{1}_{\alpha}$	3		✓	First derivative of ω^{1}_{α}	Y	8	8	(Rad)/T^2
ω^{r}_{α}	3			Inertial angular velocity of reference frame	Y	1	4,6,1, 2,7	(Rad)/T
$\dot{\omega}^{r}_{\alpha}$	3			First derivative of ω^{r}_{α}	Y	1	1,4,6,7	(Rad)/T^2
ψ^{jN}_{α}	3	✓		Rotations of n^{th} controller frame on Body j due to flexibility	Y	3	3	(Rad)
$\dot{\psi}^{jN}_{\alpha}$	3	✓		First derivative of ψ^{jN}_{α}	Y	3	3	(Rad)/T

Appendix A. Derivation of the Basic Dynamic Equations

This appendix contains the derivation of the basic dynamic equations for the UFSS Program. The first section gives details of the derivation of the governing equations for a single flexible body. Section A.2 documents the derivation of the Flexible Combining Algorithm, while Section A.3 documents the derivation of the Rigid Combining Algorithm.

A.1. Equations of Motion for a Single Flexible Body

Since only terminal bodies of the spacecraft model are assumed to be flexible, the equations for an arbitrary flexible body (Body j) are derived explicitly considering its limb (Body i). Specifically, referring to Figure A.1 the instantaneous position vector $\bar{\alpha}^j$ to an arbitrary mass element dm^j in Body j is as follows:

$$\bar{\alpha}^j = \bar{R}^r + \bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j \tag{A-1}$$

where $\bar{R}^r$ is the inertial position vector to a reference axis frame, $\bar{R}^i$, $\bar{\ell}^{ij}$, $\bar{r}^j$ and $\bar{u}^j$ are as given in Figure 2.3.

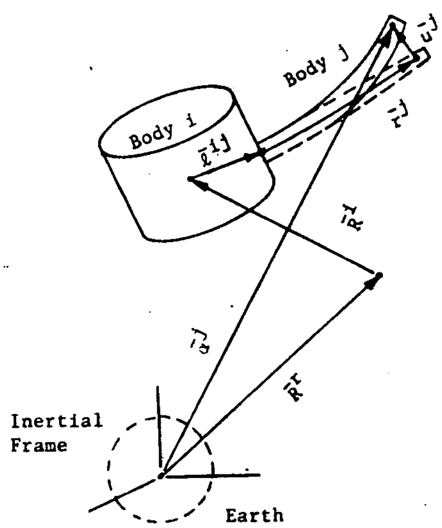


Figure A.1. Position Vector Definition for Body j

The first time derivative of $\bar{\alpha}^j$ is

$$\frac{d}{dt}(\bar{\alpha}^j) = \dot{\bar{R}}^r + \dot{\bar{R}}^i + (\bar{\omega}^r \times \bar{R}^i) + \dot{\bar{\ell}}^{ij} + (\bar{\omega}^i \times \bar{\ell}^{ij}) + (\bar{\omega}^j \times \bar{r}^j) + \dot{\bar{u}}^j + (\bar{\omega}^j \times \bar{u}^j) \tag{A-2}$$

where $\bar{\omega}^r$, $\bar{\omega}^i$ and $\bar{\omega}^j$ are the angular velocity vectors of the reference axis frame (e^r_α), the Body i fixed frame $(\underline{e}^i_\alpha)$ and the Body j fixed frame $(\underline{e}^j_\alpha)$, respectively. (The $\underline{e}^j_\alpha$ frame is assumed to define the undeformed position of Body j.)

The Kinetic energy of Body j, T^j , is formed as follows:

$$T^j = \frac{1}{2} \int_{B^j} \frac{d}{dt}(\bar{\alpha}^j) \cdot \frac{d}{dt}(\bar{\alpha}^j)\, dm^j$$

Therefore, substituting Equation (A-1) into (A-2) and performing the necessary integrations over Body j, the Kinetic energy becomes:

$$\begin{aligned} T^j = m^j \Big\{ & \frac{1}{2}\Big[\dot{\bar{R}}^r + \dot{\bar{R}}^i + (\bar{\omega}^r \times \bar{R}^i) + \dot{\bar{\ell}}^{ij} + (\bar{\omega}^i \times \bar{\ell}^{ij})\Big] \cdot \\ & \Big[\dot{\bar{R}}^r + \dot{\bar{R}}^i + (\bar{\omega}^r \times \bar{R}^i) + \dot{\bar{\ell}}^{ij} + (\bar{\omega}^i \times \bar{\ell}^{ij}) + 2\,(\bar{\omega}^j \times \bar{d}^j) \\ & + 2 \sum_{k=1}^{n_j} \dot{q}^j_k \bar{\Phi}^j_k + 2 \sum_{k=1}^{n_j} q^j_k\, (\bar{\omega}^j \times \bar{\Phi}^j_k)\Big] \\ & + \frac{1}{2}\, \bar{\omega}^j \cdot \bar{\bar{I}}^j \cdot \bar{\omega}^j + \frac{1}{2} \sum_{k,\ell=1}^{n_j} \dot{q}^j_k\, \dot{q}^j_\ell\, M^j_{k\ell} \\ & + \sum_{k=1}^{j} q^j_k\, \bar{\omega}^j \cdot \bar{\bar{N}}^j_k \cdot \bar{\omega}^j + \frac{1}{2} \sum_{k,\ell=1}^{n_j} q^j_k\, q^j_\ell\, \bar{\omega}^j \cdot \bar{\bar{E}}^j_{k\ell} \cdot \bar{\omega}^j \\ & \sum_{k=1}^{n_j} \dot{q}^j_k\, \bar{\omega}^j \cdot \bar{Y}^j_k + \sum_{k,\ell=1}^{n_j} q^j_k\, \dot{q}^j_\ell\, \bar{\omega}^j \cdot \bar{Z}^j_{k\ell} \Big\} \end{aligned}$$

where

$$m^j = \int_{B^j} dm^j = \text{mass of Body } j = \text{scalar}$$

$$\bar{d}^j = \frac{1}{m^j} \int_{B^j} \bar{r}^j \; dm^j = \sum_{\beta=1}^{3} d^j_\beta \; \underline{e}^j_\beta = \text{vector}$$

$$\bar{\phi}^j_k = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \; dm^j = \sum_{\beta=1}^{3} \phi^j_{k\beta} \; \underline{e}^j_\beta = \text{vector}$$

$$\bar{Y}^j_k = \frac{1}{m^j} \int_{B^j} \bar{r}^j \times \bar{\phi}^j_k \; dm^j = \sum_{\beta=1}^{3} Y^j_{k\beta} \; \underline{e}^j_\beta = \text{vector}$$

$$\bar{Z}^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \times \bar{\phi}^j_\ell \; dm^j = \sum_{\beta=1}^{3} Z^j_{k\ell\beta} \; \underline{e}^j_\beta = \text{vector}$$

$$M^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \bar{\phi}^j_k \cdot \bar{\phi}^j_\ell \; dm^j = \text{scalar}$$

$$\bar{\bar{I}}^j = \frac{1}{m^j} \int_{B^j} \left[\bar{\bar{\delta}}(\bar{r}^j \cdot \bar{r}^j) - \bar{r}^j \; \bar{r}^j \right] dm^j$$

$$= \sum_{\alpha,\beta=1}^{3} I^j_{\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta = \text{second order tensor}$$

$$\bar{\bar{N}}^j_k = \frac{1}{m^j} \int_{B^j} \left[\bar{\bar{\delta}}(\bar{r}^j \cdot \bar{\phi}^j_k) - \bar{r}^j \; \bar{\phi}^j_k \right] dm^j$$

$$= \sum_{\alpha,\beta=1}^{3} N^j_{k\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta = \text{second order tensor}$$

$$\bar{\bar{E}}^j_{k\ell} = \frac{1}{m^j} \int_{B^j} \left[\bar{\bar{\delta}}(\bar{\phi}^j_k \cdot \bar{\phi}^j_\ell) - \bar{\phi}^j_k \; \bar{\phi}^j_\ell \right] dm^j$$

$$= \sum_{\alpha,\beta=1}^{3} E^j_{k\ell\alpha\beta} \; \underline{e}^j_\alpha \; \underline{e}^j_\beta = \text{second order tensor}$$

$$\bar{\bar{\delta}} = \sum_{\alpha,\beta=1}^{3} \delta_{\alpha\beta} \; \underline{e}_{\alpha}^{j} \; \underline{e}_{\beta}^{j} = \text{second order unit tensor}$$

$$\delta_{\alpha\beta} = \begin{cases} 1 \, , & \beta=\alpha \\ 0 \, , & \beta\neq\alpha \end{cases} = \text{Kronëcker delta}$$

[Note: the dyad or "leaning product" formed from two vectors (such as $\bar{r}^{j} \; \bar{r}^{j}$, $\bar{r}^{j} \; \bar{\phi}_{k}^{j}$) yields a second order tensor. In particular, the following identity has been used in determining the above expression for T^{j}:

$$(\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) = (\bar{C} \cdot \bar{A}) (\bar{D} \cdot \bar{B}) - (\bar{C} \cdot \bar{B}) (\bar{D} \cdot \bar{A})$$

$$= \bar{C} \cdot [\bar{\bar{\delta}}(\bar{B} \cdot \bar{D}) - \bar{B} \; \bar{D}] \cdot \bar{A}$$

where

$\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are vectors.]

The strain energy of Body j is defined to be

$$U^{j} = \frac{1}{2} m^{j} \sum_{k,\ell=1}^{n_j} q_{k}^{j} \; q_{\ell}^{j} \; K_{k\ell}^{j}$$

where

$K_{k\ell}^{j}$ = the generalized stiffness matrix of Body j associated with the generalized coordinates q_{k}^{j} and q_{ℓ}^{j} .

Likewise, the dissipation function for Body j is defined to be

$$D^{j} = \frac{1}{2} m^{j} \sum_{k,\ell=1}^{n_j} \dot{q}_{k}^{j} \; \dot{q}_{\ell}^{j} \; V_{k\ell}^{j}$$

where

$V^j_{k\ell}$ = the generalized damping matrix of Body j associated with the generalized velocities $\dot{q}^j_k$ and $\dot{q}^j_\ell$.

Now taking as generalized coordinates for Body j the three components of $\bar{R}^i$, the p_j gimbal angles θ^j_γ $(\gamma=1,\ldots,p_j)$ and the n_j deformation coordinages q^j_k $(k=1,\ldots,n_j)$, the Lagrange Equations for Body j are as follows:

$$\left.\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial T^j}{\partial \dot{q}^j_k}\right] - \frac{\partial T^j}{\partial q^j_k} + \frac{\partial U^j}{\partial q^j_k} + \frac{\partial D^j}{\partial \dot{q}^j_k} = \mathcal{Q}^j_k \quad (k=1,\ldots,n_j)\\
&\frac{d}{dt}\left[\frac{\partial T^j}{\partial \dot{\theta}^j_\gamma}\right] - \frac{\partial T^j}{\partial \theta^j_\gamma} + \frac{\partial U^j}{\partial \theta^j_\gamma} + \frac{\partial D^j}{\partial \dot{\theta}^j_\gamma} = \textcircled{H}^j_\gamma \quad (\gamma=1,\ldots,p_j)\\
&\frac{d}{dt}\left[\frac{\partial T^j}{\partial \dot{R}^i_\alpha}\right] - \frac{\partial T^j}{\partial R^i_\alpha} + \frac{\partial U^j}{\partial R^i_\alpha} + \frac{\partial D^j}{\partial \dot{R}^i_\alpha} = \mathcal{R}^j_\alpha \quad (\alpha=1,2,3)
\end{aligned}\right\} \tag{A-3}$$

where $\mathcal{Q}^j_k$, $\textcircled{H}^j_\gamma$ and $\mathcal{R}^j_\alpha$ are the generalized forces (environmental disturbances, control forces, etc...) associated with the generalized coordinates q^j_k , θ^j_γ and R^i_γ , respectively.

A.1.1 Equation for q^j_k

Performing the indicated partial differentiation, the equation for q^j_k becomes

$$\begin{aligned}
m^j \Big\{ \Big[&\ddot{\bar{R}}^r + \ddot{\bar{R}}^i + 2(\bar{\omega}^r \times \dot{\bar{R}}^i) + (\dot{\bar{\omega}}^r \times \bar{R}^i) + \bar{\omega}^r \times (\bar{\omega}^r \times \bar{R}^i)\\
&+ \ddot{\bar{\ell}}^{ij} + 2(\bar{\omega}^i \times \dot{\bar{\ell}}^{ij}) + (\dot{\bar{\omega}}^i \times \bar{\ell}^{ij})\\
&+ \bar{\omega}^i \times (\bar{\omega}^i \times \bar{\ell}^{ij}) \Big] \cdot \bar{\Phi}^j_k + M^j_{k\ell}\, \ddot{q}^j_\ell + \dot{\bar{\omega}}^j \cdot \bar{Y}^j_k\\
&- \dot{\bar{\omega}}^j \cdot \bar{Z}^j_{k\ell}\, q^j_\ell - 2\, \bar{\omega}^j \cdot \bar{Z}^j_{k\ell}\, \dot{q}^j_\ell - \bar{\omega}^j \cdot \bar{\bar{N}}^j_k \cdot \bar{\omega}^j\\
&- \bar{\omega}^j \cdot \bar{\bar{E}}^j_{k\ell} \cdot \bar{\omega}^j\, q^j_\ell + K^j_{k\ell}\, q^j_\ell + V^j_{k\ell}\, \dot{q}^j_\ell \Big\} = \mathcal{Q}^j_k
\end{aligned}$$

(Note in the above equation and in the following, repeated subscripts are summed over their allowable range; e.g.,

$$K^j_{k\ell}\, q^j_\ell = \sum_{\ell=1}^{n_j} K^j_{k\ell}\, q^j_\ell)$$

Rearranging and writing the above in component form we have

$$m^j \Big\{ \Phi^j_{k\alpha} \Big[A^{je}_{\alpha\beta}\, \ddot{R}^r_\beta + A^{jr}_{\alpha\beta}\, \ddot{R}^i_\beta + A^{jr}_{\alpha\beta}\, (2\, \tilde{\omega}^r_{\beta\rho}\, \dot{R}^i_\rho + \dot{\tilde{\omega}}^r_{\beta\rho}\, R^i_\rho$$

$$+ \tilde{\omega}^r_{\beta\gamma}\, \tilde{\omega}^r_{\gamma\delta}\, R^i_\delta) + A^{ji}_{\alpha\beta}\, \tilde{\omega}^i_{\beta\gamma}\, \tilde{\omega}^i_{\gamma\delta}\, \ell^{ij}_\delta + A^{ji}_{\alpha\beta}\, (\ddot{\ell}^{ij}_\beta + 2\, \tilde{\omega}^i_{\beta\gamma}\, \dot{\ell}^{ij}_\gamma) \Big]$$

$$- \Phi^j_{k\alpha}\, A^{ji}_{\alpha\delta}\, \tilde{\ell}^{ij}_{\delta\beta}\, \dot{\omega}^i_\beta + M^j_{k\ell}\, \ddot{q}^j_\ell + Y^j_{k\alpha}\, \dot{\omega}^j_\alpha - H^{j1}_{k\alpha}\, \dot{\omega}^j_\alpha$$

$$- 2\, H^{j2}_{k\alpha}\, \omega^j_\alpha - N^j_{k\alpha\beta}\, \omega^j_\alpha\, \omega^j_\beta - H^{j5}_{k\alpha\beta}\, \omega^j_\alpha\, \omega^j_\beta + K^j_{k\ell}\, q^j_\ell$$

$$+ V^j_{k\ell}\, \dot{q}^j_\ell \Big\} = \mathcal{Q}^j_k$$

where:

$$H^{j1}_{k\alpha} = q^j_\ell\, Z^j_{k\ell\alpha}$$

$$H^{j2}_{k\alpha} = \dot{q}^j_\ell\, Z^j_{k\ell\alpha}$$

$$H^{j5}_{k\alpha\beta} = q^j_\ell\, E^j_{k\ell\alpha\beta}$$

Rearranging once more,

$$m^j \Big\{ M^j_{k\ell}\, \ddot{q}^j_\ell + S^{j4}_{k\alpha}\, \dot{\omega}^j_\alpha - \Phi^j_{k\alpha}\, A^{ji}_{\alpha\delta}\, \tilde{\ell}^{ij}_{\delta\beta}\, \dot{\omega}^i_\beta + \Phi^j_{k\alpha}\, A^{jr}_{\alpha\beta}\, \ddot{R}^i_\beta$$

$$+ \Phi^j_{k\alpha}\, S^{j1}_\alpha + \Phi^j_{k\alpha}\, A^{ji}_{\alpha\beta}\, S^{j2}_\beta - 2\, H^{j2}_{k\alpha}\, \omega^j_\alpha - S^{j10}_k + K^j_{k\ell}\, q^j_\ell \qquad \text{(A-4)}$$

$$+ V^j_{k\ell}\, \dot{q}^j_\ell \Big\} = \mathcal{Q}^j_k - m^j\, \Phi^j_{k\alpha}\, A^{je}_{\alpha\beta}\, \ddot{R}^r_\beta$$

where:

$$S^{j1}_{\alpha} = A^{jr}_{\alpha\sigma} \; (2 \; \tilde{\omega}^{r}_{\sigma\beta} \; \dot{R}^{i}_{\beta} + \dot{\tilde{\omega}}^{r}_{\sigma\beta} \; R^{i}_{\beta} + \tilde{\omega}^{r}_{\sigma\rho} \; \tilde{\omega}^{r}_{\rho\beta} \; R^{i}_{\beta})$$

$$+ A^{ji}_{\alpha\sigma} \; \tilde{\omega}^{i}_{\sigma\rho} \; \tilde{\omega}^{i}_{\rho\beta} \; \ell^{ij}_{\beta}$$

$$S^{j2}_{\alpha} = \ddot{\ell}^{ij}_{\alpha} + 2 \; \tilde{\omega}^{i}_{\alpha\beta} \; \dot{\ell}^{ij}_{\beta}$$

$$S^{j4}_{k\beta} = Y^{j}_{k\beta} - H^{j1}_{k\beta}$$

$$S^{j10}_{k} = (N^{j}_{k\alpha\beta} + H^{j5}_{k\alpha\beta}) \; \omega^{j}_{\alpha} \; \omega^{j}_{\beta}$$

Now, introducing the following relationships between $\bar{\omega}^{j}$, $\bar{\omega}^{i}$ and the gimbal angle derivatives,

$$\omega^{j}_{\beta} = A^{ji}_{\beta\alpha} \; \omega^{i}_{\alpha} + G^{j}_{\beta\gamma} \; \dot{\theta}^{j}_{\gamma} + A^{jjo}_{\beta\delta} \; G^{jo}_{\delta\gamma} \; \dot{\theta}^{jo}_{\gamma} \tag{A-5}$$

$$\dot{\omega}^{j}_{\beta} = A^{ji}_{\beta\alpha} \; \dot{\omega}^{i}_{\alpha} + G^{j}_{\beta\gamma} \; \ddot{\theta}^{j}_{\gamma} + S^{j5}_{\beta}$$

where

$$S^{j5}_{\beta} = -\tilde{\omega}^{j}_{\beta\gamma} \; A^{ji}_{\gamma\sigma} \; \omega^{i}_{\sigma} + \dot{G}^{j}_{\beta\gamma} \; \dot{\theta}^{j}_{\gamma} + S^{j3}_{\beta}$$

and

$$S^{j3}_{\beta} = \left[A^{jjo}_{\beta\gamma} \; \dot{G}^{jo}_{\gamma\sigma} + (A^{ji}_{\beta\gamma} \; \tilde{\omega}^{i}_{\gamma\delta} \; A^{joi(T)}_{\delta\rho} - \tilde{\omega}^{j}_{\beta\gamma} \; A^{jjo}_{\gamma\rho}) \; G^{jo}_{\rho\sigma} \right] \dot{\theta}^{jo}_{\sigma}$$

$$+ A^{jjo}_{\beta\gamma} \; G^{jo}_{\gamma\sigma} \; \ddot{\theta}^{jo}_{\sigma} \,,$$

the equation for q^{j}_{k} becomes:

$$m^{j} \Big\{ M^{j}_{k\ell} \; \ddot{q}^{j}_{\ell} + S^{j4}_{k\alpha} \; G^{j}_{\alpha\gamma} \; \ddot{\theta}^{j}_{\gamma} + S^{j4}_{k\alpha} \; A^{ji}_{\alpha\beta} \; \dot{\omega}^{i}_{\beta}$$

$$- \Phi^{j}_{k\alpha} \; A^{ji}_{\alpha\delta} \; \tilde{\ell}^{ij}_{\delta\beta} \; \dot{\omega}^{i}_{\beta} + \Phi^{j}_{k\alpha} \; A^{jr}_{\alpha\beta} \; \ddot{R}^{i}_{\beta} + S^{j4}_{k\alpha} \; S^{j5}{}_{\alpha}$$

$$+ \Phi^{j}_{k\alpha} \; S^{j7}_{\alpha} - 2 \; H^{j2}_{k\alpha} \; \omega^{j}_{\alpha} - S^{j10}_{k} + K^{j}_{k\ell} \; q^{j}_{\ell}$$

$$+ V^{j}_{k\ell} \; \dot{q}^{j}_{\ell} \Big\} = \mathcal{Q}^{j}_{k} - m^{j} \; \Phi^{j}_{k\alpha} \; A^{je}_{\alpha\beta} \; \ddot{R}^{r}_{\beta}$$

where

$$S_{\alpha}^{j7} = S_{\alpha}^{j1} + A_{\alpha\beta}^{ji} \; S_{\beta}^{j2}$$

Finally, the equation for q_k^j can be written in matrix form as follows:

$$\begin{bmatrix} B_{k\ell}^{A11} & B_{k\gamma}^{A12} & B_{k\beta}^{A13} & B_{k\beta}^{A14} \end{bmatrix} \begin{bmatrix} \ddot{q}_{\ell}^{j} \\ \ddot{\theta}_{\gamma}^{j} \\ \dot{\omega}_{\beta}^{i} \\ \ddot{R}_{\beta}^{i} \end{bmatrix} = C_k^{A1} \tag{A-6}$$

where

$$B_{k\ell}^{A11} = m^j \; M_{k\ell}^j$$

$$B_{k\gamma}^{A12} = m^j \; S_{k\alpha}^{j4} \; G_{\alpha\gamma}^j$$

$$B_{k\beta}^{A13} = m^j \; S_{k\alpha}^{j4} \; A_{\alpha\beta}^{ji} - m^j \; \Phi_{k\alpha}^j \; A_{\alpha\delta}^{ji} \; \tilde{\ell}_{\delta\beta}^{ij}$$

$$B_{k\beta}^{A14} = m^j \; \Phi_{k\alpha}^j \; A_{\alpha\beta}^{jr}$$

$$\begin{aligned} C_k^{A1} = - m^j \Big\{ & S_{k\beta}^{j4} \; S_{\beta}^{j5} + \Phi_{k\alpha}^j \; S_{\alpha}^{j7} \\ & - 2 \; H_{k\beta}^{j2} \; \omega_{\beta}^j - S_k^{j10} \\ & + K_{k\ell}^j \; q_{\ell}^j + V_{k\ell}^j \; \dot{q}_{\ell}^j \Big\} + Q_k^{je} \end{aligned}$$

It should be noted in the above equation that

$$Q_k^{je} = \mathcal{Q}_k^j - \left(- \frac{\gamma m^j}{(R^r)^3} \; \Phi_{k\alpha}^j \; A_{\alpha\beta}^{je} \; R_{\beta}^r \right)$$

that is, the gravitational force acting on Body j if Body j were located at the origin of the reference frame has been removed from the external generalized force term by equating it to the corresponding acceleration term to produce the following orbital equation

$$- \frac{\gamma m}{(R^r)^3} \Phi^j_{k\alpha} A^{je}_{\alpha\beta} R^r_\beta - m^j \Phi^j_{k\alpha} A^{je}_{\alpha\beta} \ddot{R}^r_\beta = 0 \tag{A-7}$$

or

$$\ddot{R}^r_\beta = - \frac{\gamma}{(R^r)^3} R^r_\beta$$

which is solved in the Orbit Subroutine (See Appendix B).

Thus, Q^{je}_k represents all external generalized forces associated with q^j_k except the gravity force which would act if Body j had its mass center positioned at the origin of the reference axis frame $\underline{e}^r_\alpha$ (terminus of $\bar{R}^r$).

A.1.2 Equation for R^i_α

Once again performing the indicated partial differentiation, the equation for R^i_α is as follows: (note that $\bar{R}^i = R^i_\alpha \underline{e}^r_\alpha$)

$$\begin{aligned} m^j \underline{e}^r_\alpha \cdot \Big\{ & \ddot{\bar{R}}^r + \ddot{\bar{R}}^i + 2(\bar{\omega}^r \times \dot{\bar{R}}^i) + (\dot{\bar{\omega}}^i \times \bar{R}^i) + \\ & + \bar{\omega}^r \times (\bar{\omega}^r \times \bar{R}^i) + \ddot{\bar{\ell}}^{ij} + 2(\bar{\omega}^i \times \dot{\bar{\ell}}^{ij}) + (\dot{\bar{\omega}}^i \times \bar{\ell}^{ij}) + \\ & + \bar{\omega}^i \times (\bar{\omega}^i \times \bar{\ell}^{ij}) + (\dot{\bar{\omega}}^j \times \bar{d}^j) + \bar{\omega}^j \times (\bar{\omega}^j \times \bar{d}^j) + \\ & + \bar{\Phi}^j_k \ddot{q}^j_k + 2(\bar{\omega}^j \times \bar{\Phi}^j_k) \dot{q}^j_k + (\dot{\bar{\omega}}^j \times \bar{\Phi}^j_k) q^j_k + \\ & + \bar{\omega}^j \times (\bar{\omega}^j \times \bar{\Phi}^j_k) q^j_k \Big\} = \mathcal{R}^j_\alpha . \end{aligned} \tag{A-8}$$

Rearranging and writing the above in component form we have

$$m^j \Big\{ \ddot{R}^i_\alpha + 2\, \tilde{\omega}^r_{\alpha\beta}\, \dot{R}^i_\beta + \Big[\dot{\tilde{\omega}}^r_{\alpha\beta} + \tilde{\omega}^r_{\alpha\gamma}\, \tilde{\omega}^r_{\gamma\beta} \Big] R^i_\beta$$

$$+ A^{ir(T)}_{\alpha\delta} \Big[\ddot{\ell}^{ij}_\delta + 2\, \tilde{\omega}^i_{\delta\beta}\, \dot{\ell}^{ij}_\beta - \tilde{\ell}^{ij}_{\delta\beta}\, \dot{\omega}^i_\beta + \tilde{\omega}^i_{\delta\gamma}\, \tilde{\omega}^i_{\gamma\beta}\, \ell^{ij}_\beta \Big]$$

$$+ A^{jr(T)}_{\alpha\delta} \Big[- \tilde{d}^j_{\delta\beta}\, \dot{\omega}^j_\beta + \tilde{\omega}^j_{\delta\gamma}\, \tilde{\omega}^j_{\gamma\beta}\, d^j_\beta + \Phi^{j(T)}_{\delta\ell}\, \ddot{q}^j_\ell + 2\, \tilde{\omega}^j_{\delta\beta}\, \Phi^{j(T)}_{\beta\ell}\, \dot{q}^j_\ell$$

$$- \tilde{H}^{j3}_{\delta\beta}\, \dot{\omega}^j_\beta + \tilde{\omega}^j_{\delta\gamma}\, \tilde{\omega}^j_{\gamma\beta}\, H^{j3}_\beta \Big\} = \mathcal{R}^j_\alpha - m^j\, A^{re}_{\alpha\beta}\, \ddot{R}^r_\beta$$

where $H^{j3}_\beta = q^j_k\, \Phi^j_{k\beta}$. Again introducing the relationships (A-5), the above equation becomes

$$m^j \Big\{ A^{jr(T)}_{\alpha\delta}\, \Phi^{j(T)}_{\delta\ell}\, \ddot{q}^j_\ell - m^j\, A^{jr(T)}_{\alpha\beta}\, \tilde{S}^{j6}_{\beta\delta}\, G^j_{\delta\gamma}\, \ddot{\theta}^j_\gamma$$

$$- \Big[A^{jr(T)}_{\alpha\rho}\, \tilde{S}^{j6}_{\rho\delta}\, A^{ji}_{\delta\beta} + A^{ir(T)}_{\alpha\rho}\, \tilde{\ell}^{ij}_{\rho\beta} \Big] \dot{\omega}^i_\beta + \ddot{R}^i_\alpha$$

$$+ A^{jr(T)}_{\alpha\beta} \Big[A^{jr}_{\beta\gamma} \big(2\, \tilde{\omega}^r_{\gamma\delta}\, \dot{R}^i_\delta + (\dot{\tilde{\omega}}^r_{\gamma\delta} + \tilde{\omega}^r_{\gamma\sigma}\, \tilde{\omega}^r_{\sigma\delta})\, R^i_\delta \big)$$

$$+ A^{ji}_{\beta\gamma}\, \tilde{\omega}^i_{\gamma\sigma}\, \tilde{\omega}^i_{\sigma\delta}\, \ell^{ij}_\delta + A^{ji}_{\beta\gamma} \big(\ddot{\ell}^{ij}_\gamma + 2\, \tilde{\omega}^i_{\gamma\delta}\, \dot{\ell}^{ij}_\delta \big) \Big]$$

$$+ A^{jr(T)}_{\alpha\beta} \Big[- \tilde{S}^{j6}_{\beta\gamma}\, S^{j5}_\gamma + \tilde{\omega}^j_{\beta\delta}\, \tilde{\omega}^j_{\delta\rho}\, S^{j6}_\rho + 2\, \tilde{\omega}^j_{\beta\delta}\, H^{j4}_\delta \Big\}$$

$$= \mathcal{R}^j_\alpha - m^j\, A^{re}_{\alpha\beta}\, \ddot{R}^r_\beta$$

or more simply

$$mj \Big\{ A^{jr(T)}_{\alpha\delta}\, \Phi^{j(T)}_{\delta\ell}\, \ddot{q}^j_\ell - m^j\, A^{jr(T)}_{\alpha\beta}\, \tilde{S}^{j6}_{\beta\delta}\, G^j_{\delta\gamma}\, \ddot{\theta}^j_\gamma$$

$$- \Big[A^{jr(T)}_{\alpha\rho}\, \tilde{S}^{j6}_{\rho\delta}\, A^{ji}_{\delta\beta} + A^{ir(T)}_{\alpha\rho}\, \tilde{\ell}^{ij}_{\rho\beta} \Big]\, \dot{\omega}^i_\beta + \ddot{R}^i_\alpha$$

$$+ A^{jr(T)}_{\alpha\beta} \Big[S^{j7}_\beta - \tilde{S}^{j6}_{\beta\gamma}\, S^{j5}_\gamma + \tilde{\omega}^j_{\beta\delta}\, \tilde{\omega}^j_{\delta\rho}\, S^{j6}_\rho + 2\, \tilde{\omega}^j_{\beta\delta}\, H^{j4}_\delta \Big\}$$

$$= \mathcal{R}^j_\alpha - m^j\, A^{re}_{\alpha\beta}\, \ddot{R}^r_\beta$$

where

$$S_{\alpha}^{j6} = d_{\alpha}^{j} + H_{\alpha}^{j3}$$

$$H_{\alpha}^{j4} = \dot{q}_{k}^{j} \; \Phi_{k\alpha}^{j}$$

Finally, the equation for R_{α}^{i} can be written in matrix form as follows:

$$\begin{bmatrix} B_{\alpha\ell}^{A31} & B_{\alpha\gamma}^{A32} & B_{\alpha\beta}^{A33} & B_{\alpha\beta}^{A34} \end{bmatrix} \begin{bmatrix} \ddot{q}_{\ell}^{j} \\ \ddot{\theta}_{\gamma}^{j} \\ \dot{\omega}_{\beta}^{i} \\ \ddot{R}_{\beta}^{i} \end{bmatrix} = C_{\alpha}^{A3} + A_{\alpha\beta}^{jr(T)} \; F_{\beta}^{ji} \qquad \text{(A-9)}$$

where

$$B_{\alpha\ell}^{A31} = m^{j} \; A_{\alpha\sigma}^{jr(T)} \; \Phi_{\sigma\ell}^{j(T)}$$

$$B_{\alpha\gamma}^{A32} = - m^{j} \; A_{\alpha\sigma}^{jr(T)} \; \tilde{S}_{\sigma\rho}^{j6} \; G_{\rho\gamma}^{j}$$

$$B_{\alpha\beta}^{A33} = - m^{j} \; A_{\alpha\sigma}^{jr(T)} \; \tilde{S}_{\sigma\rho}^{j6} \; A_{\rho\beta}^{ji} - m^{j} \; A_{\alpha\sigma}^{ir(T)} \; \tilde{\ell}_{\sigma\beta}^{ij}$$

$$B_{\alpha\beta}^{A34} = m^{j} \; \delta_{\alpha\beta}$$

$$C_{\alpha}^{A3} = - m^{j} \; A_{\alpha\sigma}^{jr(T)} \Big\{ S_{\sigma}^{j7} - \tilde{S}_{\sigma\beta}^{j6} \; S_{\beta}^{j5} + \tilde{\omega}_{\sigma\rho}^{j} \; \tilde{\omega}_{\rho\beta}^{j} \; S_{\beta}^{j6} + 2 \; \tilde{\omega}_{\rho\beta}^{j} \; H_{\beta}^{j4} \Big\} + R_{\alpha}^{je}$$

Once again, somewhat as in the case of the equation for q_{k}^{j},

$$R_{\alpha}^{je} = \mathcal{R}_{\alpha}^{j} - \left(- \frac{\gamma m^{j}}{(R^{r})^{3}} \; A_{\alpha\beta}^{re} \; R_{\beta}^{r} \right) - A_{\alpha\beta}^{jr(T)} \; F_{\beta}^{ji}$$

Thus, the gravitational force acting on Body j if Body j were located at the origin of the reference frame has been removed from the external generalized force term by equating it to the corresponding acceleration term to produce the orbital equation

$$-\frac{\gamma m^j}{(R^r)^3} A^{re}_{\alpha\beta} R^r_\beta - m^j A^{re}_{\alpha\beta} \ddot{R}^r_\beta = 0$$

or

$$\frac{-\gamma}{(R^r)^3} R^r_\beta = \ddot{R}^r_\beta$$

which is solved in the Orbit Subroutine.

In addition, the hinge force acting on Body j due to Body i ($\bar{F}^{ji} = F^{ji}_\alpha \, \underline{e}^j_\alpha$) has explicitly been separated from the remaining external generalized forces in order that it may be most conveniently utilized in the Flexible Combining Algorithm to be described shortly.

A.1.3 Equation for θ^j_γ

The equation for θ^j_γ can likewise be obtained by performing the "appropriate" differentiation. However, in this case T^j is an implicit function of θ^j_γ and $\dot{\theta}^j_\gamma$ only through the appearance of $\bar{\omega}^j$ and various transformation matrices occurring in the expression for T^j when written in terms of components of the vector quantities. The required differentiation and subsequent algebra is extremely long and a much simpler formulation arises by considering the torque equation about the Body j hinge:

$$\int_{B^j} \left[(\bar{r}^j + \bar{u}^j) \times \frac{d^2 \bar{\alpha}^j}{dt^2} \right] dm^j = \bar{T}^j + \bar{T}^{ji} \tag{A-10}$$

where $\bar{T}^j$ is the total external moment on Body j about the Body j hinge and $\bar{T}^{ji}$ is the hinge torque acting on Body j due to Body i. The second derivative of $\bar{\alpha}^j$ is

$$\frac{d^2 \bar{\alpha}^j}{dt^2} = \ddot{\bar{R}}^r + \ddot{\bar{R}}^i + 2(\bar{\omega}^r \times \dot{\bar{R}}^i) + (\dot{\bar{\omega}}^r \times \bar{R}^i)$$
$$+ \bar{\omega}^r \times (\bar{\omega}^r \times \bar{R}^i) + \ddot{\bar{\ell}}^{ij} + 2(\bar{\omega}^i \times \dot{\bar{\ell}}^{ij})$$
$$+ (\dot{\bar{\omega}}^i \times \bar{\ell}^{ij}) + \bar{\omega}^i \times (\bar{\omega}^i \times \bar{\ell}^{ij}) \qquad \text{(A-11)}$$
$$+ (\dot{\bar{\omega}}^j \times \bar{r}^j) + \bar{\omega}^j \times (\bar{\omega}^j \times \bar{r}^j) + \ddot{\bar{u}}^j$$
$$+ 2(\bar{\omega}^j \times \dot{\bar{u}}^j) + (\dot{\bar{\omega}}^j \times \bar{u}^j) + \bar{\omega}^j \times (\bar{\omega}^j \times \bar{u}^j)$$

Substituting (A-11) into (A-10) and performing the necessary integrations over Body j, the torque equation becomes

$$m^j \Big\{ \Big[Y^{j(T)}_{\alpha\ell} + H^{j6(T)}_{\alpha\ell} \Big] \ddot{q}^j_\ell + \Big[I^{jf}_{\alpha\beta} + H^{j10}_{\alpha\beta} + H^{j8}_{\alpha\beta} \Big] \dot{\omega}^j_\beta$$
$$- S^{j6}_{\alpha\beta} A^{ji}_{\beta\sigma} \tilde{\ell}^{ij}_{\sigma\gamma} \dot{\omega}^i_\gamma + \tilde{S}^{j6}_{\alpha\delta} A^{jr}_{\delta\beta} \ddot{R}^i_\beta + \tilde{S}^{j6}_{\alpha\beta} S^{j7}_\beta$$
$$+ \tilde{\omega}^j_{\alpha\beta} \Big[I^{jf}_{\beta\delta} + H^{j10}_{\beta\delta} + H^{j8}_{\beta\delta} \Big] \omega^j_\delta + 2 H^{j12(T)}_{\alpha\beta} \omega^j_\beta + 2 H^{j9}_{\alpha\beta} \omega^j_\beta \Big\}$$
$$= T^j_\alpha + T^{ji}_\alpha - m^j \tilde{S}^{j6}_{\alpha\beta} A^{je}_{\beta\delta} \ddot{R}^r_\delta$$

where

$$H^{j6}_{k\alpha} = q^j_\ell Z^j_{\ell k\alpha} = - H^{j1}_{k\alpha}$$

$$H^{j8}_{\alpha\beta} = H^{j11}_{\alpha\beta} + H^{j11(T)}_{\alpha\beta}$$

$$H^{j9}_{\alpha\beta} = \dot{q}^j_k q^j_\ell E^j_{k\ell\alpha\beta} = \dot{q}^j_k H^{j5}_{k\alpha\beta}$$

$$H^{j10}_{\alpha\beta} = q^j_k q^j_\ell E^j_{k\ell\alpha\beta} = q^j_k H^{j5}_{k\alpha\beta}$$

$$H^{j11}_{\alpha\beta} = q^j_\ell N^j_{\ell\alpha\beta}$$

$$H^{j12}_{\alpha\beta} = \dot{q}^j_\ell N^j_{\ell\alpha\beta}$$

Introducing the relationships (A-5), the equation for θ^j_γ becomes

$$m^j \left\{ S^{j8}_{\alpha\ell} \ddot{q}^j_\ell + S^{j9}_{\alpha\beta} G^j_{\beta\gamma} \ddot{\theta}^j_\gamma + S^{j9}_{\alpha\sigma} A^{ji}_{\sigma\beta} \dot{\omega}^i_\beta - \tilde{S}^{j6}_{\alpha\beta} A^{ji}_{\beta\sigma} \tilde{\ell}^{ij}_{\sigma\gamma} \dot{\omega}^i_\gamma + \tilde{S}^{j6}_{\alpha\delta} A^{jr}_{\delta\beta} \ddot{R}^i_\beta + S^{j9}_{\alpha\beta} S^{j5}_\beta + \tilde{S}^{j6}_{\alpha\beta} S^{j7}_\beta + \tilde{\omega}^j_{\alpha\beta} S^{j9}_{\beta\delta} \omega^j_\delta + 2 S^{j15}_{\alpha\beta} \omega^j_\beta \right\}$$

$$= T^j_\alpha + T^{ji}_\alpha - m^j \tilde{S}^{j6}_{\alpha\beta} A^{je}_{\beta\delta} \ddot{R}^r_\delta$$

where

$$S^{j8}_{\alpha\ell} = Y^{j(T)}_{\alpha\ell} + H^{j6(T)}_{\alpha\ell} = S^{j4(T)}_{\alpha\ell}$$

$$S^{j9}_{\alpha\beta} = I^{jf}_{\alpha\beta} + H^{j8}_{\alpha\beta} + H^{j10}_{\alpha\beta}$$

$$S^{j15}_{\alpha\beta} = H^{j9}_{\alpha\beta} + H^{j12(T)}_{\alpha\beta}$$

Finally, the equation for θ^j_γ can be written in matrix form as follows:

$$\begin{bmatrix} B^{A21}_{\alpha\ell} & B^{A22}_{\alpha\gamma} & B^{A23}_{\alpha\beta} & B^{A24}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{q}^j_\ell \\ \ddot{\theta}^j_\gamma \\ \dot{\omega}^i_\beta \\ \ddot{R}^i_\beta \end{bmatrix} = C^{A2}_\alpha + T^{ji}_\alpha \qquad \text{(A-12)}$$

where

$$B^{A21}_{\alpha\ell} = m^j \quad S^{j8}_{\alpha\ell}$$

$$B^{A22}_{\alpha\gamma} = m^j \quad S^{j9}_{\alpha\sigma} \quad G^j_{\sigma\gamma}$$

$$B^{A23}_{\alpha\beta} = m^j \quad S^{j9}_{\alpha\sigma} \quad A^{ji}_{\sigma\beta} - m^j \quad \tilde{S}^{j6}_{\alpha\sigma} \quad A^{ji}_{\sigma\rho} \quad \tilde{\ell}^{ij}_{\rho\beta}$$

$$B^{A24}_{\alpha\beta} = m^j \quad \tilde{S}^{j6}_{\alpha\sigma} \quad A^{jr}_{\sigma\beta}$$

$$C^{A2}_{\alpha} = - m^j \left\{ S^{j9}_{\alpha\beta} \quad S^{j5}_{\beta} + \tilde{S}^{j6}_{\alpha\beta} \quad S^{j7}_{\beta} \right.$$

$$\left. + \tilde{\omega}^j_{\alpha\sigma} \, S^{j9}_{\sigma\beta} \quad \omega^j_{\beta} + 2 \, S^{j15}_{\alpha\beta} \quad \omega^j_{\beta} \right\} + Ⓗ^{je}_{\alpha}$$

Here again,

$$Ⓗ^{je}_{\alpha} = T^j_{\alpha} - \left(- \frac{\gamma m^j}{(R^r)^3} \quad \tilde{S}^{j6}_{\alpha\beta} \quad A^{je}_{\beta\delta} \quad R^r_{\delta} \right) - T^{ji}_{\alpha} \quad .$$

Thus, the gravitational force acting on Body j if Body j were located at the origin of the reference frame has been removed from the external generalized force term by equating it to the corresponding acceleration term to produce the orbital equation

$$- \frac{\gamma m^j}{(R^r)^3} \quad \tilde{S}^{j6}_{\alpha\beta} \quad A^{je}_{\beta\delta} \quad R^r_{\delta} - m^j \quad \tilde{S}^{j6}_{\alpha\beta} \quad A^{je}_{\beta\delta} \quad \ddot{R}^r_{\delta} = 0$$

or

$$\ddot{R}^r_{\beta} = - \frac{\gamma m^j}{(R^r)^3} \quad R^r_{\beta}$$

which is solved in the Orbit Subroutine.

In addition, the hinge torque acting on Body j do to Body i ($\bar{T}^{ji} = T^{ji}_{\alpha} \quad \underline{e}^j_{\alpha}$) has explicitly been separated from the remaining torques in order that it may be most conveniently utilized in the Flexible Combining Algorithm to be described shortly.

A.1.4 Constrained Degrees of Freedom

The equations just presented for q_k^j , R_α^i and θ_γ^j have been derived assuming that all three degrees of rotational freedom exist at the gimbal hinge joining Body j and its limb, Body i. In general, there will exist fewer than three unrestrained gimbal axes and the equations must reflect this fact.

Since the transformation matrix $A_{\alpha\beta}^{ji}$ is determined from three sequential Euler rotations through the angles θ_1^j , θ_2^j and finally θ_3^j (see Section 4.1), the order in which the gimbal angle rotations are locked is important. Thus, θ_3^j is constrained first, θ_2^j is constrained next and finally, if necessary, θ_1^j is also constrained . Thus, if a single degree of freedom exists between Body i and Body j, it is θ_1^j about the first gimbal axis $\underline{e}_1^{jg}$; if two degrees of freedom exist, they must be θ_1^j and θ_2^j .

In the equations (A-6) and (A-9) for q_k^j and R_α^i respectively, constrained degrees of freedom are simply handled by replacing $G_{\alpha\gamma}^j$ by $G_{\alpha\gamma}^{j+}$ wherever it appears in the equations, where the super + implies removal of the λth column if $\theta_\lambda^j \equiv 0$.

In case of the equations for the θ_γ^j themselves, constraints are handled by introduction of $G_{\alpha\gamma}^{j+}$ wherever $G_{\alpha\gamma}^j$ apears in addition to elimination of the equations for the constrained variables. This latter operation is accomplished symbolically by introduction of a super-zero notation as follows:

$$\begin{bmatrix} B_{\delta\ell}^{A21^\circ} & B_{\delta\gamma}^{A22^\circ} & B_{\delta\beta}^{A23^\circ} & B_{\delta\beta}^{A24^\circ} \end{bmatrix} \begin{bmatrix} \ddot{q}_\ell^j \\ \ddot{\theta}_\gamma^j \\ \dot{\omega}_\beta^i \\ \ddot{R}_\beta^i \end{bmatrix} = C_\delta^{A2^\circ} + T_\delta^{ji^\circ}$$

Here: $\delta, \gamma = 1,\ldots, p_j$; $\alpha, \beta = 1, 2, 3$; $k, \ell = 1,\ldots,n_j$, where n_j is the number of flexible degrees of freedom of Body j and p_j is the number of relative rotational degrees of freedom of Body j. In case $p_j = 0$, the above equation is void and $\bar{\omega}^j$ is determined directly by $\bar{\omega}^i$ and the input values of the θ^{jo}_α through equation (4-10):

$$\omega^j_\alpha = A^{ji}_{\alpha\beta}\ \omega^i_\beta + A^{jjo}_{\alpha\beta}\ G^{jo}_{\beta\sigma}\ \dot{\theta}^{jo}_\sigma \ .$$

A.2 Details of the Flexible Combining Algorithm

The Flexible Combining Algorithm is used only when System A in a given combining operation is a single flexible body, call it Body j. Referring to Figure 5.1, assume that System B has Body i as its member of lowest level (i<j) and that System A is to be connected to System B to yield System C. (See Sections II and III for a detailed description of the system model and notation.) Assume also that Body ℓ is the lowest numbered branch of Body i in System B and that $j<\ell$.

The equations for System A are given by (A-6), (A-9) and (A-12) which are written below for reference

System A

$$\begin{bmatrix} B^{A11}_{k\ell} & B^{A12}_{k\gamma} & B^{A13}_{k\beta} & B^{A14}_{k\beta} \\ B^{A21^\circ}_{\delta\ell} & B^{A22^\circ}_{\delta\gamma} & B^{A23^\circ}_{\delta\beta} & A^{A24^\circ}_{\delta\beta} \\ B^{A31}_{\alpha\ell} & B^{A32}_{\alpha\gamma} & B^{A33}_{\alpha\beta} & B^{A34}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{q}^j_\ell \\ \ddot{\theta}^j_\gamma \\ \dot{\omega}^i_\beta \\ \ddot{R}^i_\beta \end{bmatrix} = \begin{bmatrix} C^{A1}_k \\ C^{A2^\circ}_\delta \\ C^{A3}_\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ T^{ji^\circ}_\delta \\ A^{jr(T)}_{\alpha\beta}\ F^{ji}_\beta \end{bmatrix} \qquad \text{(A-13)}$$

Let us now assume that the System B equations are of the form

System B

$$\begin{bmatrix} B^{B11}_{mn} & B^{B12}_{m\beta} & B^{B13}_{m\beta} \\ B^{B21}_{\alpha n} & B^{B22}_{\alpha\beta} & B^{B23}_{\alpha\beta} \\ B^{B31}_{\alpha n} & B^{B32}_{\alpha\beta} & B^{B33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{B1}_{m} \\ C^{B2}_{\alpha} \\ C^{B3}_{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ L^{i\hat{\ell}}_{\alpha} \\ W^{i\hat{\ell}}_{\alpha} \end{bmatrix} \tag{A-14}$$

where $L^{i\hat{\ell}}_{\alpha}$ is the total hinge torques on Body i exclusive of the torques imposed by its branches numbered $\geq \ell$ and $W^{i\hat{\ell}}$ is the total hinge forces on Body i exclusive of the forces imposed by its branches numbered $\geq \ell$.

Here, $B^{B22}_{\alpha\beta}$, $B^{B23}_{\alpha\beta}$, $B^{B32}_{\alpha\beta}$ and $B^{B33}_{\alpha\beta}$ are 3 x 3 matrices with the indices α and β running from 1 to 3; C^{B2}_{α} and C^{B3}_{α} are 3 x 1 matrices; if $\ddot{\theta}^{i\hat{\ell}}_{n}$ has r components, then B^{B11}_{mn} is r x r , $B^{B12}_{m\beta}$ and $B^{B13}_{m\beta}$ are r x 3 , $B^{B21}_{\alpha n}$ and $B^{B31}_{\alpha n}$ are 3 x r while C^{B1}_{m} is r x 1 . As used above, $\ddot{\theta}^{i\hat{\ell}}_{n}$ contains a component for every flexible and rotational degree of freedom of Body i's branches and sub-branches numbered greater than or equal to ℓ .

We shall now show that the combined System C equations have a form identical to the System B equations, namely

System C

$$\begin{bmatrix} B^{C11}_{ab} & B^{C12}_{a\beta} & B^{C13}_{a\beta} \\ B^{C21}_{\alpha b} & B^{C22}_{\alpha\beta} & B^{C23}_{\alpha\beta} \\ B^{C31}_{\alpha b} & B^{C32}_{\alpha\beta} & B^{C33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{j}}_{b} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{C1}_{a} \\ C^{C2}_{\alpha} \\ C^{C3}_{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ L^{i\hat{j}}_{\alpha} \\ W^{i\hat{j}}_{\alpha} \end{bmatrix} \tag{A-15}$$

and exactly how the elements of the coefficient matrices of (A-15) are synthesized from the coefficient matrices of (A-13) and (A-14). Having accomplished this, it will be obvious that the same synthesizing procedure can be used to combine an additional flexible body to Body i since the new System B composed of the original System B of Figure 5.1,plus Body j is governed by the Eqs.(A-15) which are identical in form to those of the original System B as given by (A-14).

The required equations governing the motion of the combined System C are as follows:

A) equations of motion for Body i translations (A-16)

B) equations of motion for Body i rotations (A-17)

C) equations of motion for Body j along those gimbal axes where it possesses a degree of freedom with respect to Body i (A-18)

D) equations of motion for the branches and sub-branches of Body i (A-19)

E) equations of motion for the flexible coordinates of Body j (A-20)

Let us first consider the equations for Body i translations. The hinge force relation for Body i is given by

$$\bar{W}^{i\hat{\ell}} = \bar{W}^{i\hat{j}} + \bar{F}^{ij} .$$

But, from Newton's Third Law

$$\bar{F}^{ij} = - \bar{F}^{ji}$$

Therefore, using (A-13) to eliminate the forces of interaction we have

$$F^{ji}_{\alpha} = A^{jr}_{\alpha\beta} \left\{ - C^{A3}_{\beta} + B^{A31}_{\beta\ell} \ddot{q}^{j}_{\ell} + B^{A32}_{\beta\gamma} \ddot{\theta}^{j}_{\gamma} + B^{A33}_{\beta\rho} \dot{\omega}^{i}_{\rho} + B^{A34}_{\beta\rho} \ddot{R}^{i}_{\rho} \right\}$$

so that, since $\bar{W}^{i\hat{\ell}} = W^{i\hat{\ell}}_{\alpha} \underline{e}^{i}_{\alpha}$, $\bar{W}^{i\hat{j}} = W^{i\hat{j}}_{\alpha} \underline{e}^{i}_{\alpha}$ and $\bar{F}^{ij} = F^{ij(T)}_{\alpha} \underline{e}^{i}_{\alpha}$ while $\bar{F}^{ji} = F^{ji(T)}_{\alpha} \underline{e}^{j}_{\alpha} = F^{ji(T)}_{\alpha} A^{ji}_{\alpha\beta} \underline{e}^{i}_{\beta}$:

$$W^{i\hat{\ell}}_{\alpha} = W^{i\hat{j}}_{\alpha} - A^{ji(T)}_{\alpha\sigma} \left\{ A^{jr}_{\sigma\beta} \left[- C^{A3}_{\beta} + B^{A31}_{\beta\ell} \ddot{q}^{j}_{\ell} + B^{A32}_{\beta\gamma} \ddot{\theta}^{j}_{\gamma} + B^{A33}_{\beta\rho} \dot{\omega}^{i}_{\rho} + B^{A34}_{\beta\rho} \ddot{R}^{i}_{\rho} \right] \right\} .$$

Thus, utilizing the last equation set of (A-14)

$$B^{B31}_{\alpha n} \ddot{\theta}^{i\hat{\ell}}_{n} + B^{B32}_{\alpha\beta} \dot{\omega}^{i}_{\beta} + B^{B33}_{\alpha\beta} \ddot{R}^{i}_{\beta} = C^{B3}_{\alpha} + W^{i\hat{j}}_{\alpha} + A^{ir}_{\alpha\beta} \left\{ C^{A3}_{\beta} - B^{A31}_{\beta\ell} \ddot{q}^{j}_{\ell} - B^{A32}_{\beta\gamma} \ddot{\theta}^{j}_{\gamma} - B^{A33}_{\beta\rho} \dot{\omega}^{i}_{\rho} - B^{A34}_{\beta\rho} \ddot{R}^{i}_{\rho} \right\}$$

or finally

$$
\begin{aligned}
& B^{B31}_{\alpha n} \ddot{\theta}^{i\hat{\ell}}_{n} + A^{ir}_{\alpha\beta} B^{A31}_{\beta\ell} \ddot{q}^{j}_{\ell} + A^{ir}_{\alpha\beta} B^{A32}_{\beta\gamma} \ddot{\theta}^{j}_{\gamma} \\
& + \left\{ B^{B32}_{\alpha\beta} + A^{ir}_{\alpha\sigma} B^{A33}_{\sigma\beta} \right\} \dot{\omega}^{i}_{\beta} + \left\{ B^{B33}_{\alpha\beta} + A^{ir}_{\alpha\sigma} B^{A34}_{\sigma\beta} \right\} \ddot{R}^{i}_{\beta} \\
& = C^{B3}_{\alpha} + A^{ir}_{\alpha\beta} C^{A3}_{\beta} + W^{i\hat{j}}_{\alpha}
\end{aligned}
\tag{A-21}
$$

Thus, the following elements of (A-15) are now determined:

$$
\begin{aligned}
B^{C31}_{\alpha b} &= \left[B^{B31}_{\alpha n} \;\middle|\; A^{ir}_{\alpha\beta} B^{A31}_{\beta\ell} \;\middle|\; A^{ir}_{\alpha\beta} B^{A32}_{\beta\gamma} \right] \\
B^{C32}_{\alpha\beta} &= B^{B32}_{\alpha\beta} + A^{ir}_{\alpha\sigma} B^{A33}_{\sigma\beta} \\
B^{C33}_{\alpha\beta} &= B^{B33}_{\alpha\beta} + A^{ir}_{\alpha\sigma} B^{A34}_{\sigma\beta} \\
C^{C3}_{\alpha} &= C^{B3}_{\alpha} + A^{ir}_{\alpha\beta} C^{A3}_{\beta}
\end{aligned}
$$

where, necessarily:

$$
\ddot{\theta}^{i\hat{j}}_{b} = \begin{bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \ddot{q}^{j}_{\ell} \\ \ddot{\theta}^{j}_{\gamma} \end{bmatrix}
\tag{A-22}
$$

with

$$
\alpha, \beta = 1,2,3
$$

$$
b = 1,2,\ldots,(r + n_j + p_j)
$$

and the column matrix $\ddot{\theta}^{j}_{\gamma}$ containing only the degree-of-freedom components at the hinge interconnection of Bodies i and j.

The equations of motion for the branches of Body i are exactly those given by the equations for $\ddot{\theta}^{i\hat{\ell}}$ in (A-14):

$$B^{B11}_{mn} \ddot{\theta}^{i\hat{\ell}}_{n} + B^{B12}_{m\beta} \dot{\omega}^{i}_{\beta} + B^{B13}_{m\beta} \ddot{R}^{i}_{\beta} = C^{B1}_{m} . \quad \text{(A-23)}$$

The equations of motion for the flexible coordinates of Body j are exactly those given by the equations for $\ddot{q}^{j}_{\ell}$ in (A-13):

$$B^{A11}_{k\ell} \ddot{q}^{j}_{\ell} + B^{A12}_{k\gamma} \ddot{\theta}^{j}_{\gamma} + B^{A13}_{k\beta} \dot{\omega}^{i}_{\beta} + B^{A14}_{k\beta} \ddot{R}^{i}_{\beta} = C^{A1}_{k} . \quad \text{(A-24)}$$

The equations of motion for Body j along those axes where it possesses a degree of freedom with respect to Body i are exactly those given by the equations for $\ddot{\theta}^{j}_{\gamma}$ in (A-13):

$$B^{A21^\circ}_{\delta\ell} \ddot{q}^{j}_{\ell} + B^{A22^\circ}_{\delta\gamma} \ddot{\theta}^{j}_{\gamma} + B^{A23^\circ}_{\delta\beta} \dot{\omega}^{i}_{\beta} + B^{A24^\circ}_{\delta\beta} \ddot{R}^{i}_{\beta} = C^{A2^\circ}_{\delta} + T^{ji^\circ}_{\delta} . \quad \text{(A-25)}$$

Therefore, the following elements of (A-15) are now also determined:

$$B^{C11}_{ab} = \begin{bmatrix} B^{B11}_{mn} & 0 & 0 \\ 0 & B^{A11}_{k\ell} & B^{A12}_{k\gamma} \\ 0 & B^{A21^\circ}_{\delta\ell} & B^{A22^\circ}_{\delta\gamma} \end{bmatrix}$$

$$B^{C12}_{a\beta} = \begin{bmatrix} B^{B12}_{m\beta} \\ B^{A13}_{k\beta} \\ B^{A23^\circ}_{\delta\beta} \end{bmatrix} ; \quad B^{C13}_{a\beta} = \begin{bmatrix} B^{B13}_{m\beta} \\ B^{A14}_{k\beta} \\ B^{A24^\circ}_{\delta\beta} \end{bmatrix}$$

$$C_a^{C1} = \begin{bmatrix} C_m^{B1} \\ C_k^{A1} \\ C_\delta^{A2^\circ} + T_\delta^{ji^\circ} \end{bmatrix}$$

Finally, consider the equations for Body i rotations. The hinge torque relation for Body i is given by

$$\bar{L}^{i\hat{\ell}} = \bar{L}^{i\hat{j}} + \bar{T}^{ij} + \bar{\ell}^{ij} \times \bar{F}^{ij}$$

But, as before

$$\bar{F}^{ij} = -\bar{F}^{ji}$$

and similarly

$$\bar{T}^{ij} = -\bar{T}^{ji}$$

Thus, from (A-13):

$$F_\alpha^{ij} = -A_{\alpha\beta}^{ir} \left\{ -C_\beta^{A3} + B_{\beta\ell}^{A31} \ddot{q}_\ell^j + B_{\beta\gamma}^{A32} \ddot{\theta}_\gamma^j + B_{\beta\rho}^{A33} \dot{\omega}_\rho^i + B_{\beta\rho}^{A34} \ddot{R}_\rho^i \right\}$$

and from (A-12): [note that all three rotational equations for Body j are required here to eliminate the interacting torques]

$$T_\alpha^{ij} = -A_{\alpha\beta}^{ij} \left\{ -C_\beta^{A2} + B_{\beta\ell}^{A21} \ddot{q}_\ell^j + B_{\beta\gamma}^{A22} \ddot{\theta}_\gamma^j + B_{\beta\rho}^{A23} \dot{\omega}_\rho^i + B_{\beta\rho}^{A24} \ddot{R}_\rho^i \right\}$$

Therefore,

$$\begin{aligned} L_\alpha^{i\hat{\ell}} = {} & L_\alpha^{i\hat{j}} \\ & - A_{\alpha\beta}^{ij} \left\{ -C_\beta^{A2} + B_{\beta\ell}^{A21} \ddot{q}_\ell^j + B_{\beta\gamma}^{A22} \ddot{\theta}_\gamma^j + B_{\beta\rho}^{A23} \dot{\omega}_\rho^i + B_{\beta\rho}^{A24} \ddot{R}_\rho^i \right\} \\ & - \tilde{\ell}_{\alpha\sigma}^{ij} A_{\sigma\beta}^{ir} \left\{ -C_\beta^{A3} + B_{\beta\ell}^{A31} \ddot{q}_\ell^j + B_{\beta\gamma}^{A32} \ddot{\theta}_\gamma^j + B_{\beta\rho}^{A33} \dot{\omega}_\rho^i + B_{\beta\rho}^{A34} \ddot{R}_\rho^i \right\} . \end{aligned}$$

Thus, using (A-14), the equations governing Body i rotations are:

$$B^{B21}_{\alpha n}\ \ddot{\theta}^{i\hat{\ell}}_{n} + B^{B22}_{\alpha\beta}\ \dot{\omega}^{i}_{\beta} + B^{B23}_{\alpha\beta}\ \ddot{R}^{i}_{\beta} = C^{B2}_{\alpha} + L^{i\hat{j}}_{\alpha}$$

$$+ A^{ji(T)}_{\alpha\beta} \left\{ C^{A2}_{\beta} - B^{A21}_{\beta\ell}\ \ddot{q}^{j}_{\ell} - B^{A22}_{\beta\gamma}\ \ddot{\theta}^{j}_{\gamma} - B^{A23}_{\beta\rho}\ \dot{\omega}^{i}_{\rho} - B^{A24}_{\beta\rho}\ \ddot{R}^{i}_{\rho} \right\}$$

$$+ \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\beta} \left\{ C^{A3}_{\beta} - B^{A31}_{\beta\ell}\ \ddot{q}^{j}_{\ell} - B^{A32}_{\beta\gamma}\ \ddot{\theta}^{j}_{\gamma} - B^{A33}_{\beta\rho}\ \dot{\omega}^{i}_{\rho} - B^{A34}_{\beta\rho}\ \ddot{R}^{i}_{\rho} \right\}$$

or, more simply:

$$B^{B21}_{\alpha n}\ \ddot{\theta}^{i\hat{\ell}}_{n} + Q^{1}_{\alpha\ell}\ \ddot{q}^{j}_{\ell} + Q^{2}_{\alpha\gamma}\ \ddot{\theta}^{j}_{\gamma} + Q^{3}_{\alpha\beta}\ \dot{\omega}^{i}_{\beta} + Q^{4}_{\alpha\beta}\ \ddot{R}^{i}_{\beta} = Q^{5}_{\alpha} + L^{i\hat{j}}_{\alpha} \qquad \text{(A-26)}$$

where:

$$Q^{1}_{\alpha\ell} = A^{ji(T)}_{\alpha\beta}\ B^{A21}_{\beta\ell} + \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\beta}\ B^{A31}_{\beta\ell}$$

$$Q^{2}_{\alpha\gamma} = A^{ji(T)}_{\alpha\beta}\ B^{A22}_{\beta\gamma} + \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\beta}\ B^{A32}_{\beta\gamma}$$

$$Q^{3}_{\alpha\beta} = B^{B22}_{\alpha\beta} + A^{ji(T)}_{\alpha\sigma}\ B^{A23}_{\sigma\beta} + \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\rho}\ B^{A33}_{\rho\beta}$$

$$Q^{4}_{\alpha\beta} = B^{B23}_{\alpha\beta} + A^{ji(T)}_{\alpha\sigma}\ B^{A24}_{\sigma\beta} + \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\rho}\ B^{A34}_{\rho\beta}$$

$$Q^{5}_{\alpha} = C^{B2}_{\alpha} + A^{ji(T)}_{\alpha\beta}\ C^{A2}_{\beta} + \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ir}_{\sigma\beta}\ C^{A3}_{\beta}$$

Thus, the remaining elements of (A-15) are now determined:

$$B^{C21}_{\alpha b} = \left[B^{B21}_{\alpha n} \;\middle|\; Q^{1}_{\alpha\ell} \;\middle|\; Q^{2}_{\alpha\gamma} \right]$$

$$B^{C22}_{\alpha\beta} = Q^{3}_{\alpha\beta} \quad ; \quad B^{C23}_{\alpha\beta} = Q^{4}_{\alpha\beta}$$

$$C^{C2}_{\alpha} = Q^{5}_{\alpha} \quad .$$

Having now demonstrated that, given the System B equations in the form (A-14), the combined System C equations are indeed in the similar form (A-15), it remains to prove that the System B equations do indeed have the form of (A-14). There are two cases to consider: first, System B is a single rigid body and; second, System B is an arbitrarily interconnected system of rigid and flexible bodies with Body i being rigid.

Considering the first case, one simply has an initialization of the System B equations. Here, the appropriate equations are the familiar Newton-Euler Equations:

$$m^i \left\{ \ddot{\bar{R}}^r + \ddot{\bar{R}}^i + 2\,\bar{\omega}^r \times \dot{\bar{R}}^i + \dot{\bar{\omega}}^r \times \bar{R}^i + \bar{\omega}^r \times (\bar{\omega}^r \times \bar{R}^i) \right\} = \bar{F}^i$$

$$\bar{\bar{I}}^i \cdot \dot{\bar{\omega}}^i + \bar{\omega}^i \times \bar{\bar{I}}^i \cdot \bar{\omega}^i = \bar{T}^i \qquad \text{(A-27)}$$

where m^i and $\bar{\bar{I}}^i$ are respectively the mass and centroidal inertia tensor (dyadic) for Body i while $\bar{F}^i$ and $\bar{T}^i$ are respectively the total external force and centroidal moment acting on Body i.

Resolving the above equations into component form and eliminating the orbital equation from the force equation, the Body i (System B equations) become:

$$m^i\, A^{ir}_{\alpha\beta} \left\{ \ddot{R}^i_\beta + 2\,\tilde{\omega}^r_{\beta\delta}\, \dot{R}^i_\delta + (\dot{\tilde{\omega}}^r_{\beta\delta} + \tilde{\omega}^r_{\beta\gamma}\, \tilde{\omega}^r_{\gamma\delta})\, R^i_\delta \right\}$$

$$= F^{ie}_\alpha + \hat{W}^{ij}_\alpha + F^{ij}_\alpha \qquad \text{(A-28)}$$

$$I^i_{\alpha\beta}\, \dot{\omega}^i_\beta + \tilde{\omega}^i_{\alpha\beta}\, I^i_{\beta\delta}\, \omega^i_\delta = T^{ie}_\alpha + \hat{L}^{ij}_\alpha + T^{ij}_\alpha + \tilde{\ell}^{ij}_{\alpha\beta}\, F^{ij}_\beta$$

where F^{ie}_α is the total external force acting on Body i exclusive of all hinge forces and the gravitational force Body i would experience if it were located at the terminus of $\bar{R}^r$, while T^{ie}_α is the total external centroidal moment acting on Body i exclusive of all hinge-produced moments.

Thus, in this first case the System B specification [i.e., specification of the coefficient matrices of (A-14)] is as follows:

$$B^{B22}_{\alpha\beta} = I^{i}_{\alpha\beta}$$

$$B^{B33}_{\alpha\beta} = m^{j} \; A^{ir}_{\alpha\beta}$$

$$B^{B23}_{\alpha\beta} = B^{A33}_{\alpha\beta} = 0$$

All the remaining sub-matrices of B^{B} are void since $\ddot{\theta}^{i\hat{\ell}}_{n}$ has zero components (m,n=o).

In addition,

$$C^{B2}_{\alpha} = - \tilde{\omega}^{i}_{\alpha\beta} \; I^{i}_{\beta\gamma} \; \omega^{i}_{\delta} + T^{ie}_{\alpha}$$

$$C^{B3}_{\alpha} = - m^{i} \; A^{ir}_{\alpha\beta} \left\{ 2 \, \tilde{\omega}^{r}_{\beta\delta} \; \dot{R}^{i}_{\delta} + (\dot{\tilde{\omega}}^{r}_{\beta\delta} + \tilde{\omega}^{r}_{\beta\gamma} \; \tilde{\omega}^{r}_{\gamma\delta}) \; R^{i}_{\delta} \right\} + F^{ie}_{\alpha}$$

while C^{B1}_{m} is void.

Considering now the second possible case; that is that System B is an arbitrarily interconnected system of rigid and flexible bodies with Body i being rigid, it is sufficient to demonstrate a second combining algorithm which synthesizes the equations for two arbitrary systems of interconnected bodies when System A is not a single flexible body. This algorithm will henceforth be denoted as the "Rigid Combining Algorithm", and its output will now be shown to have the required form of (A-15).

A.3 Details of the Rigid Combining Algorithm

Consider the combining of two systems (Systems A and B) to form a third system (System C) as shown in Figure 3.3. Here, Body j of level (N+1) is the lowest leveled body of System A and Body s is the lowest numbered branch of Body j. Body i of level N is the lowest leveled body of System B and Body ℓ is the lowest numbered branch of Body i. Since the highest numbered branches of a given limb are connected first by the Sequencing Algorithm specification, it is necessarily true that $j<\ell$.

Here, System B is with no loss of generality identical to the System B of Figure (A-2), hence its equations must be identical to those given by (A-14), namely

System B

$$\begin{bmatrix} B^{B11}_{mn} & B^{B12}_{m\beta} & B^{B13}_{m\beta} \\ B^{B21}_{\alpha n} & B^{B22}_{\alpha\beta} & B^{B23}_{\alpha\beta} \\ B^{B31}_{\alpha n} & B^{B32}_{\alpha\beta} & B^{B33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{i\hat{\ell}}_{n} \\ \dot{\omega}^{i}_{\beta} \\ \ddot{R}^{i}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{B1}_{m} \\ C^{B2}_{\alpha} \\ C^{B3}_{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ L^{i\hat{\ell}}_{\alpha} \\ W^{i\hat{\ell}}_{\alpha} \end{bmatrix} \quad \text{(A-29)}$$

In addition, System A must have a similar description and interpretation

System A

$$\begin{bmatrix} B^{A11}_{k\ell} & B^{A12}_{k\beta} & B^{A13}_{k\beta} \\ B^{A21}_{\alpha\ell} & B^{A22}_{\alpha\beta} & B^{A23}_{\alpha\beta} \\ B^{A31}_{\alpha\ell} & B^{A32}_{\alpha\beta} & B^{A33}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \ddot{\theta}^{j\hat{s}}_{\ell} \\ \dot{\omega}^{j}_{\beta} \\ \ddot{R}^{j}_{\beta} \end{bmatrix} = \begin{bmatrix} C^{A1}_{k} \\ C^{A2}_{\alpha} \\ C^{A3}_{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ L^{j\hat{s}}_{\alpha} \\ W^{j\hat{s}}_{\alpha} \end{bmatrix} \quad \text{(A-30)}$$

Here, $B^{A22}_{\alpha\beta}$, $B^{A23}_{\alpha\beta}$, $B^{A32}_{\alpha\beta}$ and $B^{A33}_{\alpha\beta}$ are 3 x 3 matrices; C^{A2}_{α} and C^{A3}_{α} are 3 x 1; if $\ddot{\theta}^{j\hat{s}}_{\ell}$ has M components, then $B^{A11}_{k\ell}$ is M x M, $B^{A12}_{k\beta}$ and $B^{A13}_{k\beta}$ are M x 3, $B^{A21}_{\alpha\ell}$ and $B^{A31}_{\alpha\ell}$ are 3 x M while C^{A1}_{k} is M x 1. Once again, $\ddot{\theta}^{j\hat{s}}_{\ell}$ contains a component for every flexible and rotational degree of freedom of Body j's branches and sub-branches numbered greater than or equal to s .

We will now show that the combined System C equations have the form of (A-15) as desired. Specifically, the required equations governing the motion of the combined System C are as follows:

A) equations of motion for Body i translations

B) equations of motion for Body i rotations

C) equations of motion for the branches and sub-branches of Body i

D) equations of motion for Body j along those gimbal axes where it possesses a degree of freedom with respect to Body i

E) equations of motion for the branches and sub-branches of Body j

Before deriving the above equations, certain relationships must be established. Since Body j is a rigid body it is true by definition that

$$\bar{R}^j = \bar{R}^i + \bar{\ell}^{ij} - \bar{\ell}^{ji} \ .$$

Thus, taking a first time derivative,

$$\dot{\bar{R}}^j + \bar{\omega}^r \times \bar{R}^j = \dot{\bar{R}}^i + \bar{\omega}^r \times \bar{R}^i + \dot{\bar{\ell}}^{ij} + \bar{\omega}^i \times \bar{\ell}^{ij} - \dot{\bar{\ell}}^{ji} - \bar{\omega}^j \times \bar{\ell}^{ji}$$

or

$$\dot{\bar{R}}^j = \dot{\bar{R}}^i + \bar{\omega}^r \times (\bar{R}^i - \bar{R}^j) + \dot{\bar{\ell}}^{ij} + \bar{\omega}^i \times \bar{\ell}^{ij} - \dot{\bar{\ell}}^{ji} - \bar{\omega}^j \times \bar{\ell}^{ji} \ .$$

In component form (and substituting for $\bar{R}^i - \bar{R}^j$) one obtains the recursive relationship for $\dot{R}^j_\alpha$ in terms of $\dot{R}^i_\alpha$:

$$\begin{aligned} \dot{R}^j_\alpha = \dot{R}^i_\alpha &+ A^{ir(T)}_{\alpha\beta}\, \dot{\ell}^{ij}_\beta - A^{jr(T)}_{\alpha\beta}\, \dot{\ell}^{ji}_\beta \\ &+ \left\{ A^{ir(T)}_{\alpha\beta}\, \tilde{\omega}^i_{\beta\gamma} - \tilde{\omega}^r_{\alpha\beta}\, A^{ir(T)}_{\beta\gamma} \right\} \ell^{ij}_\gamma \\ &- \left\{ A^{jr(T)}_{\alpha\beta}\, \tilde{\omega}^j_{\beta\gamma} - \tilde{\omega}^r_{\alpha\beta}\, A^{jr(T)}_{\beta\gamma} \right\} \ell^{ji}_\gamma \end{aligned} \qquad \text{(A-31)}$$

Taking a second time derivative in vector form:

$$\ddot{\bar{R}}^j + \bar{\omega}^r \times \dot{\bar{R}}^j = \ddot{\bar{R}}^i + \bar{\omega}^r \times \dot{\bar{R}}^i + \dot{\bar{\omega}}^r \times (\bar{R}^i - \bar{R}^j) + \bar{\omega}^r \times (\dot{\bar{R}}^i - \dot{\bar{R}}^j)$$

$$+ \bar{\omega}^r \times \left[\bar{\omega}^r \times (\bar{R}^i - \bar{R}^j)\right] + \ddot{\bar{\ell}}^{ij} + \bar{\omega}^i \times \dot{\bar{\ell}}^{ij} + \dot{\bar{\omega}}^i \times \bar{\ell}^{ij}$$

$$+ \bar{\omega}^i \times \dot{\bar{\ell}}^{ij} + \bar{\omega}^i \times (\bar{\omega}^i \times \bar{\ell}^{ij}) - \ddot{\bar{\ell}}^{ji} - \bar{\omega}^j \times \dot{\bar{\ell}}^{ji} - \dot{\bar{\omega}}^j \times \bar{\ell}^{ji}$$

$$- \bar{\omega}^j \times \dot{\bar{\ell}}^{ji} - \bar{\omega}^j \times (\bar{\omega}^j \times \bar{\ell}^{ji})$$

or, in component form:

$$\ddot{R}^j_\alpha = \ddot{R}^i_\alpha + P^{10}_{\alpha\beta}\,\dot{\omega}^i_\beta + P^{19}_{\alpha\beta}\,\dot{\omega}^j_\beta + P^{16}_\alpha \tag{A-32}$$

where

$$P^{10}_{\alpha\beta} = -A^{ir(T)}_{\alpha\gamma}\,\tilde{\ell}^{ij}_{\gamma\beta}$$

$$P^{19}_{\alpha\beta} = A^{jr(T)}_{\alpha\gamma}\,\tilde{\ell}^{ji}_{\gamma\beta}$$

$$P^{16}_\alpha = A^{ir(T)}_{\alpha\beta}\left\{\ddot{\ell}^{ij}_\beta + 2\,\tilde{\omega}^i_{\beta\delta}\,\dot{\ell}^{ij}_\delta + \tilde{\omega}^i_{\beta\delta}\,\tilde{\omega}^i_{\delta\gamma}\,\ell^{ij}_\gamma\right\}$$

$$- A^{jr(T)}_{\alpha\beta}\left\{\ddot{\ell}^{ji}_\beta + 2\,\tilde{\omega}^j_{\beta\delta}\,\dot{\ell}^{ji}_\delta + \tilde{\omega}^j_{\beta\delta}\,\tilde{\omega}^j_{\delta\gamma}\,\ell^{ji}_\gamma\right\}$$

$$+\left\{\dot{\tilde{\omega}}^r_{\alpha\beta} + \tilde{\omega}^r_{\alpha\gamma}\,\tilde{\omega}^r_{\gamma\beta}\right\}\left\{R^i_\beta - R^j_\beta\right\} + 2\,\tilde{\omega}^r_{\alpha\beta}\left\{\dot{R}^i_\beta - \dot{R}^j_\beta\right\}$$

Let us first consider the equations for Body i translations. The hinge force relation for Body i is once again given by

$$\bar{W}^{i\hat{\ell}} = \bar{W}^{i\hat{j}} + \bar{F}^{ij}$$

so that, using the relation $\bar{F}^{ij} = -\bar{F}^{ji}$, one has

$$W^{i\hat{\ell}}_\alpha = W^{i\hat{j}}_\alpha - A^{ji(T)}_{\alpha\beta}\,F^{ji}_\beta \;.$$

But, since Body s is the lowest numbered branch of Body i,it follows that

$$W_\alpha^{j\hat{s}} = F_\alpha^{ji}$$

so that

$$W_\alpha^{i\hat{\ell}} = W_\alpha^{i\hat{j}} - A_{\alpha\beta}^{ji(T)} W_\beta^{j\hat{s}} \tag{A-33}$$

Thus, using (A-29) and (A-30),

$$B_{\alpha n}^{B31} \ddot{\theta}_n^{i\hat{\ell}} + B_{\alpha\beta}^{B32} \dot{\omega}_\beta^i + B_{\alpha\beta}^{B33} \ddot{R}_\beta^i - C_\alpha^{B3} = W_\alpha^{i\hat{j}}$$
$$- A_{\alpha\beta}^{ji(T)} \left\{ B_{\beta\ell}^{A31} \ddot{\theta}_\ell^{j\hat{s}} + B_{\beta\delta}^{A32} \dot{\omega}_\delta^j + B_{\beta\delta}^{A33} \ddot{R}_\delta^j - C_\alpha^{A3} \right\}$$

Substituting for $\dot{\omega}_\delta^j$ and $\ddot{R}_\delta^j$ one finds (noting that here $P_\alpha^{17} \equiv S_\alpha^{j5}$)

$$B_{\alpha n}^{B31} \ddot{\theta}_n^{i\hat{\ell}} + B_{\alpha\beta}^{B32} \dot{\omega}_\beta^i + B_{\alpha\beta}^{B33} \ddot{R}_\beta^i - C_\alpha^{B3} = W_\alpha^{i\hat{j}}$$
$$- A_{\alpha\beta}^{ji(T)} \left\{ B_{\beta\ell}^{A31} \ddot{\theta}_\ell^{j\hat{s}} + B_{\beta\delta}^{A32} \left[A_{\delta\gamma}^{ji} \dot{\omega}_\gamma^i + G_{\delta\gamma}^{j+} \ddot{\theta}_\gamma^j + P_\delta^{17} \right] \right.$$
$$\left. + B_{\beta\delta}^{A33} \left[\ddot{R}_\delta^i + P_{\delta\gamma}^{10} \dot{\omega}_\gamma^i + P_{\delta\gamma}^{19} \left(A_{\gamma\sigma}^{ji} \dot{\omega}_\sigma^i + G_{\gamma\sigma}^{j+} \ddot{\theta}_\sigma^j + P_\gamma^{17} \right) + P_\delta^{16} \right] - C_\beta^{A3} \right\}$$

or

$$B_{\alpha n}^{B31} \ddot{\theta}_n^{i\hat{\ell}} + A_{\alpha\beta}^{ji(T)} B_{\beta\ell}^{A31} \ddot{\theta}_\ell^{j\hat{s}} + A_{\alpha\beta}^{ji(T)} (B_{\beta\delta}^{A32} G_{\delta\gamma}^{j+} + B_{\beta\delta}^{A33} P_{\delta\rho}^{19} G_{\rho\gamma}^{j+}) \ddot{\theta}_\gamma^j$$
$$+ \left[B_{\alpha\beta}^{B32} + A_{\alpha\delta}^{ji(T)} \left(B_{\delta\gamma}^{A32} A_{\gamma\beta}^{ji} + B_{\delta\gamma}^{A33} P_{\gamma\beta}^{10} + B_{\delta\gamma}^{A33} P_{\gamma\rho}^{19} A_{\rho\beta}^{ji} \right) \right] \dot{\omega}_\beta^i$$
$$+ \left[B_{\alpha\beta}^{B33} + A_{\alpha\delta}^{ji(T)} B_{\delta\beta}^{A33} \right] \ddot{R}_\beta^i = W_\alpha^{i\hat{j}} + C_\alpha^{B3} - A_{\alpha\beta}^{ji(T)} \left[B_{\beta\delta}^{A32} P_\delta^{17} \right.$$
$$\left. + B_{\beta\delta}^{A33} \left(P_{\delta\rho}^{19} P_\rho^{17} + P_\delta^{16} \right) - C_\beta^{A3} \right]$$

or

$$B^{B31}_{\alpha n}\ \ddot{\theta}^{i\hat{\ell}}_{n} + A^{ji(T)}_{\alpha\beta}\ B^{A31}_{\beta\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + P^{12}_{\alpha\beta}\ G^{j+}_{\beta\gamma}\ \ddot{\theta}^{j}_{\gamma} + \left\{ B^{B32}_{\alpha\beta} + A^{ji(T)}_{\alpha\delta}\ B^{A33}_{\delta\gamma}\ P^{10}_{\gamma\beta} \right.$$

$$\left. + P^{12}_{\alpha\delta}\ A^{ji}_{\delta\beta} \right\} \dot{\omega}^{i}_{\beta} + \left\{ B^{B33}_{\alpha\beta} + A^{ji(T)}_{\alpha\delta}\ B^{A33}_{\delta\beta} \right\} \ddot{R}^{i}_{\beta} = C^{B3}_{\alpha} - P^{12}_{\alpha\beta}\ P^{17}_{\beta} \tag{A-34}$$

$$+ A^{ji(T)}_{\alpha\beta} \left\{ C^{A3}_{\beta} - B^{A33}_{\beta\delta}\ P^{16}_{\delta} \right\} + W^{ij}_{\alpha}$$

where

$$P^{12}_{\alpha\delta} = A^{ji(T)}_{\alpha\beta} \left\{ B^{A32}_{\beta\delta} + B^{A33}_{\beta\gamma}\ P^{19}_{\gamma\delta} \right\}$$

Finally then, the following elements of (A-15) are determined:

$$B^{C31}_{\alpha b} = \left[B^{B31}_{\alpha n} \;\middle|\; A^{ji(T)}_{\alpha\beta}\ B^{A31}_{\beta\ell} \;\middle|\; P^{12}_{\alpha\beta}\ G^{j+}_{\beta\gamma} \right]$$

$$B^{C32}_{\alpha\beta} = B^{B32}_{\alpha\beta} + A^{ji(T)}_{\alpha\delta}\ B^{A33}_{\delta\gamma}\ P^{10}_{\gamma\beta} + P^{12}_{\alpha\delta}\ A^{ji}_{\delta\beta}$$

$$B^{C33}_{\alpha\beta} = B^{B33}_{\alpha\beta} + A^{ji(T)}_{\alpha\delta}\ B^{A33}_{\delta\beta}$$

$$C^{C3}_{\alpha} = C^{B3}_{\alpha} - P^{12}_{\alpha\beta}\ P^{17}_{\beta} + A^{ji(T)}_{\alpha\beta} \left\{ C^{A3}_{\beta} - B^{A33}_{\beta\delta}\ P^{16}_{\delta} \right\} .$$

The equations for the branches and sub-branches of Body i are exactly those given by the equations for $\ddot{\theta}^{j\hat{\ell}}$ in (A-29):

$$B^{B11}_{mn}\ \ddot{\theta}^{i\hat{\ell}}_{n} + B^{B12}_{m\beta}\ \dot{\omega}^{i}_{\beta} + B^{B13}_{m\beta}\ \ddot{R}^{i}_{\beta} = C^{B1}_{m} \tag{A-35}$$

The equations for the branches and sub-branches of Body j are determined from those for $\ddot{\theta}^{j\hat{s}}$ in (A-30):

$$B^{A11}_{k\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A12}_{k\beta}\ \dot{\omega}^{j}_{\beta} + B^{A13}_{k\beta}\ \ddot{R}^{j}_{\beta} = C^{A1}_{k}$$

Substituting for $\dot{\omega}^{j}_{\beta}$ and $\ddot{R}^{j}_{\beta}$ one finds

$$B^{A11}_{k\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A12}_{k\beta}\left\{A^{ji}_{\beta\delta}\ \dot{\omega}^{i}_{\delta} + G^{j+}_{\beta\gamma}\ \ddot{\theta}^{j}_{\gamma} + P^{17}_{\beta}\right\}$$

$$+ B^{A13}_{k\delta}\left\{\ddot{R}^{i}_{\delta} + P^{10}_{\delta\gamma}\ \dot{\omega}^{i}_{\gamma} + P^{19}_{\delta\gamma}\left(A^{ji}_{\gamma\sigma}\ \dot{\omega}^{i}_{\sigma} + G^{j+}_{\gamma\sigma}\ \ddot{\theta}^{j}_{\sigma} + P^{17}_{\gamma}\right) + P^{16}_{\delta}\right\} = C^{A1}_{k}$$

or

$$B^{A11}_{k\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + P^{1}_{k\gamma}\ \ddot{\theta}^{j}_{\gamma} + P^{4}_{k\beta}\ \dot{\omega}^{i}_{\beta} + B^{A13}_{k\beta}\ \ddot{R}^{i}_{\beta} = P^{13}_{k} \qquad \text{(A-36)}$$

where

$$P^{1}_{k\gamma} = \left\{B^{A12}_{k\beta} + B^{A13}_{k\delta}\ P^{19}_{\delta\beta}\right\} G^{j+}_{\beta\gamma}$$

$$P^{4}_{k\beta} = \left\{B^{A12}_{k\delta} + B^{A13}_{k\sigma}\ P^{19}_{\sigma\delta}\right\} A^{ji}_{\delta\beta} + B^{A13}_{k\delta}\ P^{10}_{\delta\beta}$$

$$P^{13}_{k} = C^{A1}_{k} - \left\{B^{A12}_{k\beta} + B^{A13}_{k\delta}\ P^{19}_{\delta\beta}\right\} P^{17}_{\beta} - B^{A13}_{k\delta}\ P^{16}_{\delta}$$

The equations of motion for Body j along those gimbal axes where it possesses a degree of freedom with respect to Body i are obtained from the equations for $\dot{\omega}^{j}_{\beta}$ in (A-30):

$$B^{A21}_{\alpha\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A22}_{\alpha\beta}\ \dot{\omega}^{j}_{\beta} + B^{A23}_{\alpha\beta}\ \ddot{R}^{j}_{\beta} = C^{B2}_{\alpha} + L^{j\hat{s}}_{\alpha}$$

But,

$$L^{j\hat{s}}_{\alpha} = T^{ji}_{\alpha} + \tilde{\ell}^{ji}_{\alpha\beta}\ F^{ji}_{\beta}$$

and as above

$$F^{ji}_{\beta} = W^{j\hat{s}}_{\beta}$$

so that

$$L^{j\hat{s}}_{\alpha} = T^{ji}_{\alpha} + \tilde{\ell}^{ji}_{\alpha\beta}\ W^{j\hat{s}}_{\beta}\ . \qquad \text{(A-37)}$$

Substituting from (A-30):

$$B^{A21}_{\alpha\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A22}_{\alpha\beta}\ \dot{\omega}^{j}_{\beta} + B^{A23}_{\alpha\beta}\ \ddot{R}^{j}_{\beta} - C^{A2}_{\alpha} =$$

$$T^{ji}_{\alpha} + \tilde{\ell}^{ji}_{\alpha\sigma}\left\{ B^{A31}_{\sigma\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A32}_{\sigma\beta}\ \dot{\omega}^{j}_{\beta} + B^{A33}_{\sigma\beta}\ \ddot{R}^{j}_{\beta} - C^{B3}_{\sigma} \right\}$$

or

$$\left\{ B^{A21}_{\alpha\ell} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A31}_{\sigma\ell} \right\} \ddot{\theta}^{j\hat{s}}_{\ell} + \left\{ B^{A22}_{\alpha\beta} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A32}_{\sigma\beta} \right\} \dot{\omega}^{j}_{\beta}$$

$$+ \left\{ B^{A23}_{\alpha\beta} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A33}_{\sigma\beta} \right\} \ddot{R}^{j}_{\beta} = T^{ji}_{\alpha} - \tilde{\ell}^{ji}_{\alpha\sigma}\ C^{A3}_{\sigma} + C^{A2}_{\alpha} \ .$$

Substituting for $\dot{\omega}^{j}_{\beta}$ and $\ddot{R}^{j}_{\beta}$, one obtains

$$\left\{ B^{A21}_{\alpha\ell} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A31}_{\sigma\ell} \right\} \ddot{\theta}^{j\hat{s}}_{\ell} + \left\{ B^{A22}_{\alpha\beta} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A32}_{\sigma\beta} \right\} \left\{ A^{ji}_{\beta\gamma}\ \dot{\omega}^{i}_{\gamma}\right.$$

$$\left. + G^{j+}_{\beta\gamma}\ \ddot{\theta}^{j}_{\gamma} + P^{17}_{\beta} \right\} + \left\{ B^{A23}_{\alpha\delta} - \tilde{\ell}^{ji}_{\alpha\rho}\ B^{A33}_{\rho\delta} \right\} \left\{ \ddot{R}^{i}_{\delta} + P^{10}_{\delta\gamma}\ \dot{\omega}^{i}_{\gamma}\right.$$

$$\left. + P^{19}_{\delta\gamma}\left(A^{ji}_{\gamma\sigma}\ \dot{\omega}^{i}_{\sigma} + G^{j+}_{\gamma\sigma}\ \ddot{\theta}^{j}_{\sigma} + P^{17}_{\gamma} \right) + P^{16}_{\delta} \right\} = T^{ji}_{\alpha}$$

$$- \tilde{\ell}^{ji}_{\alpha\sigma}\ C^{A3}_{\sigma} + C^{A2}_{\alpha} \ .$$

Rearranging,

$$P^{2}_{\alpha\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + P^{3}_{\alpha\gamma}\ \ddot{\theta}^{j}_{\gamma} + P^{5}_{\alpha\beta}\ \dot{\omega}^{i}_{\beta} + P^{6}_{\alpha\beta}\ \ddot{R}^{i}_{\beta} = P^{14}_{\alpha} \qquad \text{(A-38)}$$

where:

$$P^{2}_{\alpha\ell} = \left\{ B^{A21}_{\alpha\ell} - \tilde{\ell}^{ji}_{\alpha\sigma}\ B^{A31}_{\sigma\ell} \right\}^{\circ}$$

$$P^{3}_{\alpha\gamma} = \left\{ P^{20}_{\alpha\beta}\ G^{j+}_{\beta\gamma} \right\}^{\circ}$$

$$P^{20}_{\alpha\beta} = B^{A22}_{\alpha\beta} + B^{A23}_{\alpha\delta}\ P^{19}_{\delta\beta} - \tilde{\ell}^{ji}_{\alpha\sigma}\left\{ B^{A32}_{\sigma\beta} + B^{A33}_{\sigma\rho}\ P^{19}_{\rho\beta} \right\}$$

$$P^{5}_{\alpha\beta} = \left\{ P^{20}_{\alpha\delta} \; A^{ji}_{\delta\beta} + P^{21}_{\alpha\delta} \; P^{10}_{\delta\beta} \right\}^{\circ}$$

$$P^{21}_{\alpha\delta} = B^{A23}_{\alpha\delta} - \tilde{\ell}^{ji}_{\alpha\rho} \; B^{A33}_{\rho\delta}$$

$$P^{6}_{\alpha\beta} = \left\{ P^{21}_{\alpha\beta} \right\}^{\circ}$$

$$P^{14}_{\alpha} = \left\{ - P^{20}_{\alpha\beta} \; P^{17}_{\beta} - P^{21}_{\alpha\beta} \; P^{16}_{\beta} + C^{A2}_{\alpha} - \tilde{\ell}^{ji}_{\alpha\sigma} \; C^{A3}_{\sigma} + T^{ji}_{\alpha} \right\}^{\circ}$$

Where the super zero indicates retention of only the rows for which $\theta^{j}_{\gamma} \not\equiv 0$; i.e., the λth row is eliminated if $\theta^{j}_{\lambda} \equiv 0$. Thus, using (A-35), (A-36) and (A-38), the following elements of (A-15) are also determined:

$$B^{C11}_{pq} = \begin{bmatrix} B^{B11}_{mn} & 0 & 0 \\ 0 & B^{A11}_{k\ell} & P^{1}_{k\gamma} \\ 0 & P^{2}_{\alpha\ell} & P^{3}_{\alpha\gamma} \end{bmatrix}$$

$$B^{C12}_{p\beta} = \begin{bmatrix} B^{B12}_{m\beta} \\ P^{4}_{k\beta} \\ P^{5}_{\alpha\beta} \end{bmatrix}$$

$$B^{C13}_{p\beta} = \begin{bmatrix} B^{B13}_{m\beta} \\ B^{A13}_{k\beta} \\ P^{6}_{\alpha\beta} \end{bmatrix} \quad ; \quad C^{C1}_{p} = \begin{bmatrix} C^{B1}_{m} \\ P^{13}_{k} \\ P^{14}_{\alpha} \end{bmatrix}$$

Finally, consider the equations for Body i rotations. Once again,

$$\bar{L}^{i\hat{\ell}} = \bar{L}^{i\hat{j}} + \bar{T}^{ij} + \ell^{-ij} \times \bar{F}^{ij}$$

But,

$$\bar{T}^{ij} = -\bar{T}^{ji} \quad \text{and} \quad \bar{F}^{ij} = -\bar{F}^{ji}$$

Therefore,

$$\bar{L}^{i\hat{\ell}} = \bar{L}^{i\hat{j}} - \bar{T}^{ji} - \bar{\ell}^{ij} \times \bar{F}^{ji}$$

and using the relationships

$$\bar{F}^{ji} = \bar{W}^{j\hat{s}}$$

$$\bar{T}^{ji} = \bar{L}^{j\hat{s}} - \bar{\ell}^{ji} \times \bar{F}^{ji}$$

one has

$$\bar{L}^{i\hat{\ell}} = \bar{L}^{i\hat{j}} - \bar{L}^{j\hat{s}} + \bar{\ell}^{ji} \times \bar{W}^{j\hat{s}} - \bar{\ell}^{ij} \times \bar{W}^{j\hat{s}} \qquad \text{(A-39)}$$

Substituting from (A-29) and (A-30),

$$\begin{aligned}
& B^{B21}_{\alpha n}\ \ddot{\theta}^{i\hat{\ell}}_{n} + B^{B22}_{\alpha\beta}\ \dot{\omega}^{i}_{\beta} + B^{B23}_{\alpha\beta}\ \ddot{R}^{i}_{\beta} - C^{B2}_{\alpha} = L^{i\hat{j}}_{\alpha} \\
& - A^{ji(T)}_{\alpha\sigma} \left\{ B^{A21}_{\sigma\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A22}_{\sigma\beta}\ \dot{\omega}^{j}_{\beta} + B^{A23}_{\sigma\beta}\ \ddot{R}^{j}_{\beta} - C^{A2}_{\sigma} \right\} \\
& + A^{ji(T)}_{\alpha\sigma}\ \tilde{\ell}^{ji}_{\sigma\rho} \left\{ B^{A31}_{\rho\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A32}_{\rho\beta}\ \dot{\omega}^{j}_{\beta} + B^{A33}_{\rho\beta}\ \ddot{R}^{j}_{\beta} - C^{A3}_{\rho} \right\} \\
& - \tilde{\ell}^{ij}_{\alpha\sigma}\ A^{ji(T)}_{\sigma\rho} \left\{ B^{A31}_{\rho\ell}\ \ddot{\theta}^{j\hat{s}}_{\ell} + B^{A32}_{\rho\beta}\ \dot{\omega}^{j}_{\beta} + B^{A33}_{\rho\beta}\ \ddot{R}^{j}_{\beta} - C^{A3}_{\rho} \right\}
\end{aligned}$$

or,

$$B^{B21}_{\alpha n} \ddot{\theta}^{i\hat{\ell}}_{n} + \left\{ P^{18}_{\alpha\rho} B^{A31}_{\rho\ell} + A^{ji(T)}_{\alpha\rho} B^{A21}_{\rho\ell} \right\} \ddot{\theta}^{j\hat{s}}_{\ell}$$

$$+ \left\{ P^{18}_{\alpha\rho} B^{A32}_{\rho\beta} + A^{ji(T)}_{\alpha\rho} B^{A22}_{\rho\beta} \right\} \dot{\omega}^{j}_{\beta} + B^{B22}_{\alpha\beta} \dot{\omega}^{i}_{\beta} + \left\{ P^{18}_{\alpha\rho} B^{A33}_{\rho\beta} + A^{ji(T)}_{\alpha\rho} B^{A23}_{\rho\beta} \right\} \ddot{R}^{j}_{\beta}$$

$$+ B^{B23}_{\alpha\beta} \ddot{R}^{i}_{\beta} = L^{i\hat{j}}_{\alpha} + C^{B2}_{\alpha} + P^{18}_{\alpha\rho} C^{A3}_{\rho} + A^{ji(T)}_{\alpha\sigma} C^{A2}_{\sigma}$$

where

$$P^{18}_{\alpha\beta} = \tilde{\ell}^{ij}_{\alpha\gamma} A^{ji(T)}_{\gamma\beta} - A^{ji(T)}_{\alpha\gamma} \tilde{\ell}^{ji}_{\gamma\beta} .$$

Substituting now for $\dot{\omega}^{j}_{\beta}$ and $\ddot{R}^{j}_{\beta}$ one has

$$B^{B21}_{\alpha n} \ddot{\theta}^{i\hat{\ell}}_{n} + P^{7}_{\alpha\ell} \ddot{\theta}^{j\hat{s}}_{\ell} + \left\{ P^{18}_{\alpha\rho} B^{A32}_{\rho\beta} + A^{ji(T)}_{\alpha\rho} B^{A22}_{\rho\beta} \right\} \left\{ A^{ji}_{\beta\gamma} \dot{\omega}^{i}_{\gamma} \right.$$

$$\left. + G^{j+}_{\beta\gamma} \ddot{\theta}^{j}_{\gamma} + P^{17}_{\beta} \right\} + B^{B22}_{\alpha\beta} \dot{\omega}^{i}_{\beta} + \left\{ P^{18}_{\alpha\rho} B^{A33}_{\rho\beta} + A^{ji(T)}_{\alpha\rho} B^{A23}_{\rho\beta} \right\} \left\{ \ddot{R}^{i}_{\beta} \right.$$

$$\left. + P^{10}_{\beta\gamma} \dot{\omega}^{i}_{\gamma} + P^{19}_{\beta\gamma} \left(A^{ji}_{\gamma\sigma} \dot{\omega}^{i}_{\sigma} + G^{j+}_{\gamma\sigma} \ddot{\theta}^{j}_{\sigma} + P^{17}_{\gamma} \right) + P^{16}_{\beta} \right\} + B^{B23}_{\gamma\beta} \ddot{R}^{i}_{\beta}$$

$$= L^{i\hat{j}}_{\alpha} + C^{B2}_{\alpha} + P^{18}_{\alpha\rho} C^{A3}_{\rho} + A^{ji(T)}_{\alpha\sigma} C^{A2}_{\sigma}$$

where

$$P^{7}_{\alpha\ell} = P^{18}_{\alpha\rho} B^{A31}_{\rho\ell} + A^{ji(T)}_{\alpha\rho} B^{A21}_{\rho\ell}$$

Simplifying the above,

$$B^{B21}_{\alpha n} \ddot{\theta}^{i\hat{\ell}}_{n} + P^{7}_{\alpha\ell} \ddot{\theta}^{j\hat{s}}_{\ell} + P^{8}_{\alpha\gamma} \ddot{\theta}^{j}_{\gamma} + \left\{ P^{11}_{\alpha\gamma} A^{ji}_{\gamma\beta} - P^{9}_{\alpha\gamma} P^{10}_{\gamma\beta} \right.$$

$$\left. + B^{B22}_{\alpha\beta} \right\} \dot{\omega}^{i}_{\beta} + \left\{ B^{B23}_{\alpha\beta} - P^{9}_{\alpha\beta} \right\} \ddot{R}^{i}_{\beta} = C^{B2}_{\alpha} + P^{9}_{\alpha\beta} P^{16}_{\beta} \tag{A-40}$$

$$- P^{11}_{\alpha\beta} P^{17}_{\beta} + A^{ji(T)}_{\alpha\beta} C^{A2}_{\beta} + P^{18}_{\alpha\beta} C^{A3}_{\beta} + L^{i\hat{j}}_{\alpha}$$

where

$$P^{8}_{\alpha\gamma} = P^{11}_{\alpha\beta} \, G^{j+}_{\beta\gamma}$$

$$P^{9}_{\alpha\gamma} = - P^{18}_{\alpha\beta} \, B^{A33}_{\beta\gamma} - A^{ji(T)}_{\alpha\beta} \, B^{A23}_{\beta\gamma}$$

$$P^{11}_{\alpha\beta} = P^{18}_{\alpha\gamma} \, B^{A32}_{\gamma\beta} + A^{ji(T)}_{\alpha\gamma} \, B^{A22}_{\gamma\beta} - P^{9}_{\alpha\gamma} \, P^{19}_{\gamma\beta}$$

Thus, the final elements of (A-15) are now determined:

$$B^{C21}_{\alpha q} = \left[B^{B21}_{\alpha n} \;\middle|\; P^{7}_{\alpha \ell} \;\middle|\; P^{8}_{\alpha\gamma} \right]$$

$$B^{C22}_{\alpha\beta} = - P^{9}_{\alpha\gamma} \, P^{10}_{\gamma\beta} + P^{11}_{\alpha\gamma} \, A^{ji}_{\gamma\beta} + B^{B22}_{\alpha\beta}$$

$$B^{C23}_{\alpha\beta} = B^{B23}_{\alpha\beta} - P^{9}_{\alpha\beta}$$

$$C^{C2}_{\alpha} = C^{B2}_{\alpha} + P^{9}_{\alpha\beta} \, P^{16}_{\beta} - P^{11}_{\alpha\beta} \, P^{17}_{\beta} + A^{ji(T)}_{\alpha\beta} + P^{18}_{\alpha\beta} \, C^{A3}_{\beta} \quad .$$

It has now been demonstrated that given the two arbitrary systems A and B of Figure C.3 with equations of the form (A-29) and (A-30), the equations of the combined System C are indeed of the form (A-15) which allows the combined system to be used as either a System A or a System B in a subsequent combining operation. Finally, if System A or System B consist of a single rigid body, the initialization procedure is identical to that given by (A-28) and the relations immediately following these equations.

Appendix B. Derivation of Environmental Disturbances

B.1 Derivation of Gravity Nominal Forces and Torques

For a spacecraft in a Kepler orbit, the magnitude of the gravitational force at any point varies inversely as the square of the distance to the orbited body's center. Referring to Figure 2.3, one finds that the instantaneous force acting on a differential mass element of the terminal flexible body is given by

$$d\,\bar{F}^j = -\frac{\gamma\,\bar{\rho}^j}{(\rho^j)^3}\,dm^j \tag{B-1}$$

where γ is the orbited body's gravitational constant,

$$\bar{\rho}^j = \bar{R}^r + \bar{R}_f^j \,, \tag{B-2}$$

$$\bar{R}_f^j = \bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j \,, \tag{B-3}$$

and

$$(\rho^j) = (\bar{\rho}^j \cdot \bar{\rho}^j)^{1/2} \,.$$

In order to determine $(\rho^j)^3$, one first calculates $\bar{\rho}^j \cdot \bar{\rho}^j$ as

$$\bar{\rho}^j \cdot \bar{\rho}^j = (R^r)^2 + 2\,\bar{R}^r \cdot \bar{R}_f^j + (R_f^j)^2$$

so that

$$\bar{\rho}^j \cdot \bar{\rho}^j = (R^r)^2\left[1 + 2\,\frac{\bar{R}^r \cdot \bar{R}_f^j}{(R^r)^2} + \left(\frac{R_f^j}{R^r}\right)^2\right]$$

and

$$\frac{1}{(\rho^j)^3} = \frac{1}{(R^r)^3}\left[1 + 2\,\frac{\bar{R}^r \cdot \bar{R}_f^j}{(R^r)^2} + \left(\frac{R_f^j}{R^r}\right)^2\right]^{-3/2} \,. \tag{B-4}$$

A nominal value of (R_f^j / R^r) is 10^{-7} so that $(R_f^j / R^r)^2$ is small compared to the first two terms in (B-4) and can be neglected. Expanding the resulting expression in a binominal series and retaining only first order terms yields

$$\frac{1}{(\rho^j)^3} = \frac{1}{(R^r)^3} \left[1 - 3 \frac{\bar{R}^r \cdot \bar{R}_f^j}{(R^r)^2} \right] . \tag{B-5}$$

Substituting (B-5) into (B-1), one obtains the following expression for the differential force acting on dm^j :

$$d\,\bar{F}^j = - \frac{\gamma}{(R^r)^3} \left[1 - 3 \frac{\bar{R}^r \cdot \bar{R}_f^j}{(R^r)^2} \right] (\bar{R}^r + \bar{R}_f^j)\, dm^j . \tag{B-6}$$

Substituting the proper expression for $\bar{R}_f^j$ from (B-1) one has

$$d\,\bar{F}^j = - \frac{\gamma}{(R^r)^3} \left[1 - 3 \frac{\bar{R}^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)}{(R^r)^2} \right] (\bar{R}^r + \bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\, dm^j .$$

Expanding the above expression,

$$d\,\bar{F}^j = - \frac{\gamma\, \bar{R}^r}{(R^r)^3}\, dm^j - \frac{\gamma}{(R^r)^3} (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\, dm^j$$

$$+ \frac{3\gamma}{(R^r)^5} \left[\bar{R}^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j) \right] (\bar{R}^r + \bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\, dm^j .$$

Neglecting flexible displacements compared to the dimensions of the spacecraft and neglecting terms of order higher than $(R^r)^{-3}$, one obtains the final differential gravity force to be used in subsequent calculations:

$$\begin{aligned} d\bar{F}^j = & - \frac{\gamma\, \bar{R}^r}{(R^r)^3} - \frac{\gamma}{(R^r)^3} (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\, dm^j \\ & + \frac{3\,\gamma}{(R^r)^5} \left[\bar{R}^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j) \right] \bar{R}^r\, dm^j . \end{aligned} \tag{B-7}$$

If Body j is rigid, then $\bar{u}^j \equiv 0$ and the total gravity force is

$$\bar{F}^j = - \frac{\gamma m^j \bar{R}^r}{(R^r)^3} + \bar{F}^{jG}$$

where the first term is the force Body j would experience if it were located at the terminous of $\bar{R}^r$; this term is combined with the proper acceleration term from the dynamics equation for Body j and solved in the Orbital Subroutine as described previously in Section IV. Thus,

$$\bar{F}^{jG} = \int_{B^j} d\,\bar{F}^j$$

$$= - \gamma m^j \frac{\bar{R}^j}{(R^r)^3} + \frac{3 \gamma m^j}{(R^r)^3} (\bar{R}^r \cdot \bar{R}^j)\, \bar{R}^r \ .$$

But,

$$\bar{R}^r = R^r \underline{a}^r \quad \text{(see Orbit Subroutine)}$$

Therefore

$$\bar{F}^{jG} = - \gamma m^j \frac{\bar{R}^j}{(R^r)^3} + \frac{3 \gamma m^j}{(R^r)^3} (\underline{a}^r \cdot \bar{R}^j)\, \underline{a}^r$$

or

$$F^{jG}_{\alpha} = - \frac{\gamma m^j}{(R^r)^3} \left[A^{jr}_{\alpha\beta} R^j_{\beta} - 3 A^{je}_{\alpha\beta} a^r_{\beta} \left(R^{j(T)}_{\gamma} A^{re}_{\gamma\delta} a^r_{\delta} \right)\right] \tag{B-8}$$

The gravity torque on the rigid Body j can be expressed as

$$\bar{T}^{jG} = \int_{B^j} \bar{\zeta}^j \times d\,\bar{F}^j \tag{B-9}$$

where $\bar{\zeta}^j$ is the position vector of an arbitrary mass point in Body j.

Substituting the gravitational force in the form

$$d\,\bar{F}^j = -\frac{\gamma}{(R^r)^3}\left(\bar{R}^r + \bar{R}^j + \bar{\zeta}^j\right) dm^j$$

$$+ \frac{3\,\gamma}{(R^r)^5}\left[\bar{R}^r \cdot (\bar{R}^j + \bar{\zeta}^j)\right]\bar{R}^r \; dm^j$$

into Equation (119),

$$\bar{T}^{jG} = \frac{3\,\gamma}{(R^r)^5}\int_{B^j} (\bar{\zeta}^j \times \bar{R}^r)\,(\bar{R}^r \cdot \bar{\zeta}^j)\; dm^j$$

or

$$\bar{T}^{jG} = \frac{3\,\gamma}{(R^r)^5}\int_{B^j} (\bar{R}^r \cdot \bar{\zeta}^j)\,\bar{\zeta}^j \; dm^j \times \bar{R}^r$$

Subtracting zero from the above equation,

$$\bar{T}^{jG} = \frac{3\,\gamma}{(R^r)^5}\int_{B^j} \bar{\zeta}^j\,(\bar{\zeta}^j \cdot \bar{R}^r)\; dm^j \times \bar{R}^r$$

$$- \frac{3\,\gamma}{(R^r)^5}\int_{B^j} \zeta^{j2}\;\bar{R}^r\; dm^j \times \bar{R}^r$$

or, with $\bar{\bar{I}}$ = the unit dyadic,

$$\bar{T}^{jG} = \frac{3\,\gamma}{(R^r)^5}\,\bar{R}^r \times \int_{B^j} \{\bar{\bar{I}}\,(\bar{\zeta}^j \cdot \bar{\zeta}^j)\cdot \bar{R}^r - \bar{\zeta}^j\,\bar{\zeta}^j \cdot \bar{R}^r\}\; dm^j$$

$$= \frac{3\,\gamma}{(R^r)^5}\,\bar{R}^r \times \int_{B^j} \{\bar{\bar{I}}\,(\bar{\zeta}^j \cdot \bar{\zeta}^j) - \bar{\zeta}^j\;\bar{\zeta}^j\; dm^j\} \cdot \bar{R}^r$$

$$= \frac{3\,\gamma}{(R^r)^5}\,\bar{R}^r \times \bar{\bar{I}}^j \cdot \bar{R}^r$$

where $\bar{\bar{I}}^j$ is the centroidal inertia dyadic of Body j.

Finally,

$$T_{\alpha}^{jG} = \frac{3\gamma}{(R^r)^3} A_{\alpha\beta}^{je} \tilde{a}_{\beta\delta}^{r} A_{\delta\gamma}^{je^{(T)}} I_{\gamma\sigma}^{j} A_{\sigma\rho}^{je} a_{\rho}^{r} \tag{B-10}$$

where $I_{\alpha\beta}^{j}$ is the Body j inertia matrix with components expressed in Body j axes with origin at the center of mass.

B.2 Derivation of Gravity Generalized Forces

If Body j is a flexible body, then the general expression for the force on an element of mass, dm^j, is

$$\begin{aligned} d\,\bar{F}^j = & -\frac{\gamma\,\bar{R}^r}{(R^r)^3}\, dm^j - \frac{\gamma}{(R^r)^3} (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\, dm^j \\ & + \frac{3\gamma}{(R^r)^3} \{\underline{a}^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + \bar{u}^j)\}\, \underline{a}^r\, dm^j \end{aligned} \tag{B-11}$$

Therefore, the virtual work done by $d\bar{F}^j$ acting on dm^j due to the virtual displacement

$$\delta\bar{R}^i + \delta\bar{u}^j + \delta\bar{\theta}^j \times (\bar{r}^j + \bar{u}^j)$$

is

$$\delta W^{jg} = \int_{B^j} d\bar{F}^j \cdot \left[\delta\bar{R}^i + \delta\bar{u}^j + \delta\bar{\theta}^j \times (\bar{r}^j + \bar{u}^j)\right] \tag{B-12}$$

where

$$\delta\bar{R}^i = \delta R_{\alpha}^{i}\, \underline{e}_{\alpha}^{r}$$

$$\delta\bar{u}^j = \delta q_k^j\, \phi_{k\alpha}^{j}\, \underline{e}_{\alpha}^{j}$$

$$\delta\bar{\theta}^j = \delta\theta_{\alpha}^{j}\, \underline{e}_{\alpha}^{jg}$$

also,

$$\delta W^{jg} = R_{\alpha}^{jg} \; \delta R_{\alpha}^{i} + Q_{k}^{jg} \; \delta q_{k}^{j} + Q\,H_{\alpha}^{jg} \; \delta \theta_{\alpha}^{j}$$

so that

$$R_{\alpha}^{jg} = \int_{B^j} d\,\bar{F}^j \cdot \underline{e}_{\alpha}^{r}$$

$$Q_{k}^{jg} = \int_{B^j} d\,\bar{F}^j \cdot \underline{e}_{\alpha}^{j} \; \phi_{k\alpha}^{j}$$

$$Q\,H_{\alpha}^{jg} = \int_{B^j} d\,\bar{F}^j \cdot \{\underline{e}_{\alpha}^{jg} \times (\bar{r}^j + \bar{u}^j)\}$$

where R_{α}^{jg}, Q_{k}^{jg} and $Q\,H_{\alpha}^{jg}$ are the gravity generalized forces associated with the generalized coordinates R_{α}^{i}, q_{k}^{j} and θ_{α}^{j} respectively.

Thus,

$$R_{\alpha}^{jg} = -\frac{\gamma \; m^j}{(R^r)^3} \; \bar{R}_{\alpha}^{r} \cdot \underline{e}_{\alpha}^{r} - \frac{\gamma \; m^j}{(R^r)^3} (\bar{R}^i + \bar{\ell}^{ij} + \bar{d}^j + \bar{H}^{j3}) \cdot \underline{e}_{\alpha}^{r}$$

$$+ \frac{3\,\gamma m^j}{(R^r)^3} \{\underline{a}^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{d}^j + \bar{H}^{j3})\}\underline{a}^r \cdot \underline{e}_{\alpha}^{r}$$

or

$$R_{\alpha}^{jg} = -\frac{\gamma \; m^j}{(R^r)^3} \; A_{\alpha\beta}^{re} \; R_{\beta}^{r} - \frac{\gamma \; m^j}{(R^r)^3} \{R_{\alpha}^{j} + A_{\alpha\beta}^{jr(T)} \; H_{\beta}^{j3}\}$$

$$+ \frac{3\gamma \; m^j}{(R^r)^3} \; A_{\alpha\beta}^{re} \; a_{\beta}^{r} \; \{(R_{\sigma}^{j(T)} \; A_{\sigma\rho}^{re} + H_{\sigma}^{j3(T)} \; A_{\sigma\rho}^{je}) \; a_{\rho}^{r}\}$$

Finally, subtracting off the orbital term as described in Appendix C,

$$R_{\alpha}^{jG} = -\frac{\gamma \; m^j}{(R^r)^3} \{R_{\alpha}^{j} + A_{\alpha\beta}^{jr(T)} \; H_{\beta}^{j3}\}$$

$$+ \frac{3\,\gamma m^j}{(R^r)^3} \; A_{\alpha\beta}^{re} \; a_{\beta}^{r} \; \{(R_{\sigma}^{j(T)} \; A_{\sigma\rho}^{re} + H_{\sigma}^{j3(T)} \; A_{\sigma\rho}^{je}) \; a_{\rho}^{r}\} \qquad \text{(B-13)}$$

Consider now Q_k^{jq} :

$$Q_k^{jg} = -\frac{\gamma\ m^j}{(R^r)^3}\ \bar{R}^r \cdot \underline{e}_\alpha^j\ \phi_{k\alpha}^j - \frac{\gamma}{(R^r)^3}\int_{B^j} (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + q_\ell^j\ \bar{\phi}_\ell^j) \cdot \bar{\phi}_k^j\ dm^j$$

$$+ \frac{3\ \gamma}{(R^r)^3}\int_{B^j} \{a^r \cdot (\bar{R}^i + \bar{\ell}^{ij} + \bar{r}^j + q_\ell^j\ \bar{\phi}_\ell^j)\}\ \{\underline{a}^r \cdot \bar{\phi}_k^j\}\ dm^j$$

or

$$Q_k^{jg} = -\frac{\gamma\ m^j}{(R^r)^3}\ \phi_{k\alpha}^j\ A_{\alpha\beta}^{je}\ R_\beta^r - \frac{\gamma\ m^j}{(R^r)^3}\ \bar{\eta}^j \cdot \bar{\Phi}_k^j$$

$$-\frac{\gamma}{(R^r)^3}\int_{B^j} \bar{r}^j \cdot \bar{\phi}_k^j\ dm^j - \frac{\gamma}{(R^r)^3}\int_{B^j} \bar{\phi}_k^j \cdot \bar{\phi}_\ell^j\ dm^j\ q_\ell^j$$

$$+\frac{3\gamma\ m^j}{(R^r)^3}\ (\underline{a}^r \cdot \bar{\eta}^j)\ (\underline{a}^r \cdot \bar{\bar{\Phi}}_k^j) + \frac{3\ \gamma}{(R^r)^3}\ \underline{a}^r \cdot \int_{B^j} \bar{r}^j\ \bar{\phi}_k^j\ dm^j \cdot \underline{a}^r$$

$$+\frac{3\ \gamma}{(R^r)^3}\ \bar{a}^r \cdot \int_{B^j} \bar{\phi}_k^j\ \bar{\phi}_\ell^j\ dm^j \cdot \underline{a}^r\ q_\ell^j$$

where $\bar{\eta}^j = \bar{R}^i + \bar{\ell}^{ij}$

Defining the following quantities:

$$\bar{\bar{B}}_k^j = \frac{1}{m^j}\int_{B^j} \bar{r}^j\ \bar{\phi}_k^j\ dm^j$$

$$\bar{\bar{C}}_{k\ell}^j = \frac{1}{m^j}\int_{B^j} \bar{\phi}_k^j\ \bar{\phi}_\ell^j\ dm^j = C_{k\ell\alpha\beta}^j\ \underline{e}_\alpha^j\ \underline{e}_\beta^j$$

$$D_k^j = \frac{1}{m^j}\int_{B^j} \bar{r}^j \cdot \bar{\phi}_k^j\ dm^j = \text{scalar},$$

one has

$$Q_k^{jg} = -\frac{\gamma\ m^j}{(R^r)^3}\ \Phi_{k\alpha}^j\ A_{\alpha\beta}^{je}\ R_\beta^r\ -\ \frac{\gamma}{(R^r)^3}\ \{\bar{\Phi}_{k\alpha}^j\ A_{\alpha\beta}^{jr}\ \eta_\beta^j$$

$$+\ D_k^j\ +\ M_{k\ell}^j\ q_\ell^j\ -\ 3\ \bar{\Phi}_{k\alpha}^j\ A_{\alpha\beta}^{je}\ a_\beta^r\ (a_\sigma^{r(T)}\ A_{\sigma\rho}^{re(T)}\ \eta_\rho^j)$$

$$-\ 3\ b_{k\alpha}^j\ A_{\alpha\beta}^{je}\ a_\beta^r\ -\ 3\ c_{k\ell}^j\ q_\ell^j\}$$

where

$$b_{k\gamma}^j = a_\sigma^{r(T)}\ A_{\sigma\delta}^{je(T)}\ B_{k\delta\gamma}^j$$

$$c_{k\ell}^j = a_\sigma^{r(T)}\ A_{\sigma\alpha}^{je(T)}\ C_{k\ell\alpha\beta}^j\ A_{\beta\rho}^{je}\ a_\rho^r\ .$$

Finally, subtracting off the orbital term as described in Appendix C:

$$Q_k^{jG} = -\frac{\gamma\ m^j}{(R^r)^3}\{\Phi_{k\alpha}^j\ A_{\alpha\beta}^{jr}\ \eta_\beta^j\ +\ D_k^j\ +\ M_{k\ell}^j\ q_\ell^j\ -\ 3\ c_{k\ell}^j\ q_\ell^j$$

(B-14)

$$-\ 3\ \Phi_{k\alpha}^j\ A_{\alpha\beta}^{je}\ a_\beta^r\ (a_\sigma^{r(T)}\ A_{\sigma\rho}^{re(T)}\ \eta_\rho^j)\ -\ 3\ b_{k\alpha}^j\ A_{\alpha\beta}^{je}\ a_\beta^r\}$$

Consider now $Q\,H^{jg}$:

$$Q\,H_\alpha^{jg} = \underline{e}_\alpha^{jg} \cdot \int_{B^j} \{(\bar{r}^j + \bar{u}^j) \times d\,\bar{F}^j\}$$

$$= G_{\alpha\beta}^{j(T)}\ \underline{e}_\beta^j \cdot \int_{B^j} \{(\bar{r}^j + \bar{u}^j) \times d\,\bar{F}^j\}$$

But,

$$\int_{B^j} \{(\bar{r}^j + \bar{u}^j) \times d\,\bar{F}^j\} = -\frac{\gamma\ m^j}{(R^r)^3}\ \bar{S}^{j6} \times (\bar{R}^r + \bar{\eta}^j) + \frac{3\gamma\ m^j}{(R^r)^3}(\bar{S}^{j6} \times \underline{a}^r)(\underline{a}^r \cdot \bar{\eta}^j)$$

$$+ \frac{3\ \gamma}{(R^r)^3} \int_{B^j} \left[(\bar{r}^j + q^j_\ell\ \bar{\phi}^j_\ell) \times \underline{a}^r\right]\left[(\bar{r}^j + q^j_k\ \bar{\phi}^k_k) \cdot \underline{a}^r\right] dm^j \ .$$

Using the relation $(\bar{a} \times \bar{b})\ (\bar{c} \cdot \bar{d}) = -\ \bar{b} \times (\bar{a}\ \bar{c} \cdot \bar{d})$, the last term above becomes

$$-\frac{3\ \ \gamma}{(R^r)^3}\ \underline{a}^r \times \int_{B^j} \left[\bar{r}^j\ \bar{r}^j + \bar{r}^j\ \bar{\phi}^j_k\ q^j_k + q^j_\ell\ \bar{\phi}^j_\ell\ \bar{r}^j + q^j_\ell\ \ \bar{\phi}^j_\ell\ \bar{\phi}^j_k\ q^j_k\right] dm^j \cdot \underline{a}^r$$

$$= -\frac{3\gamma\ m^j}{(R^r)^3}\ \underline{a}^r \times \{\bar{\bar{A}}^j + \bar{\bar{B}}^j_k\ q^j_k + \bar{\bar{B}}^{j(T)}_\ell\ q^j_\ell + q^j_k\ \bar{\bar{C}}^j_{k\ell}\ q^j_\ell\} \cdot \underline{a}^r$$

$$= -\frac{3\gamma\ m^j}{(R^r)^3}\ \{-\ \underline{a}^r \times \bar{\bar{I}}^{jf} \cdot \underline{a}^r + \underline{a}^r \times \bar{b}^j_k\ q^j_k + \underline{a}^r \times \bar{S}^{j11}_k\ q^j_k + \underline{a}^r \times \bar{S}^{j13}_\ell\ q^j_\ell\}$$

where: $m^j\ \bar{\bar{I}}^{jf}$ is the Body j inertia dyadic about its hinge

$$\bar{S}^{j11}_k = \bar{\bar{B}}^{j(T)}_k \cdot \underline{a}^r = B^j_{k\alpha\beta}\ A^{je}_{\beta\sigma}\ a^r_\sigma\ \underline{e}^j_\alpha$$

$$\bar{S}^{j13}_\ell = q^j_k\ \bar{S}^{j12}_{k\ell}\ ;\quad \bar{S}^{j12}_{k\ell} = \bar{\bar{C}}^j_{k\ell} \cdot \underline{a}^r$$

Finally then, since $(H)^{jg}_\alpha = \left(G^{j(T)}\right)^{-1}_{\alpha\beta}\ Q(H)^{jg}_\beta$,

$$(H)^{jg}_\alpha = -\frac{\gamma\ m^j}{(R^r)^3}\ \underline{e}^j_\alpha \cdot (\bar{S}^{j6} \times \bar{R}^r) - \frac{\gamma\ m^j}{(R^r)^3}\ \underline{e}^j_\alpha \cdot (\bar{S}^{j6} \times \bar{\eta}^j)$$

$$+ \frac{3\gamma\ m^j}{(R^r)^3}\ \underline{e}^j_\alpha \cdot (\bar{S}^{j6} \times \underline{a}^r)\ (\underline{a}^r \cdot \bar{\eta}^j)$$

$$-\frac{3\gamma\ m^j}{(R^r)^3}\ \underline{e}^j_\alpha \cdot \underline{a}^r \times \{-\ \bar{\bar{I}}^{jf} \cdot \underline{a}^r + (\bar{b}^j_k + \bar{S}^{j11}_k + \bar{S}^{j13}_k)\ q^j_k\}$$

or

$$
Ⓗ^{jg}_{\alpha} = -\frac{\gamma\ m^{j}}{(R^{r})^{3}}\ \tilde{S}^{j6}_{\alpha\beta}\ A^{je}_{\beta\delta}\ R^{r}_{\delta} - \frac{\gamma\ m^{j}}{(R^{r})^{3}}\ \tilde{S}^{j6}_{\alpha\beta}\ A^{jr}_{\beta\delta}\ \eta^{j}_{\delta}
$$

$$
+ \frac{3\ \gamma m^{j}}{(R^{r})^{3}}\ \tilde{S}^{j6}_{\alpha\beta}\ A^{je}_{\beta\delta}\ a^{r}_{\delta}\ (\eta^{j(T)}_{\sigma}\ A^{re}_{\sigma\rho}\ a^{r}_{\rho})
$$

$$
- \frac{3\ \gamma m^{j}}{(R^{r})^{3}}\ A^{je}_{\alpha\beta}\ \tilde{a}^{r}_{\beta\gamma}\ A^{je(T)}_{\gamma\delta}\ \{-\ I^{jf}_{\gamma\sigma}\ A^{je}_{\sigma\rho}\ a^{r}_{\rho} + (b^{j(T)}_{\delta k} + S^{j11(T)}_{\delta k} + S^{j13(T)}_{\delta k})q^{j}_{k}\}.
$$

Thus, subtracting off the orbital term as described in Appendix C:

$$
Ⓗ^{jG}_{\alpha} = \frac{3\gamma\ m^{j}}{(R^{r})^{3}}\ A^{je}_{\alpha\beta}\ \tilde{a}^{r}_{\beta\gamma}\ A^{je(T)}_{\gamma\delta}\ I^{jf}_{\delta\sigma}\ A^{je}_{\sigma\rho}\ a^{r}_{\rho}
$$

$$
- \frac{\gamma\ m^{j}}{(R^{r})^{3}}\ \{\tilde{S}^{j6}_{\alpha\beta}\ A^{jr}_{\beta\delta}\ \eta^{j}_{\delta} - 3\ \tilde{S}^{j6}_{\alpha\beta}\ A^{je}_{\beta\delta}\ a^{r}_{\delta}\ (\eta^{j(T)}_{\sigma}\ A^{re}_{\sigma\rho}\ a^{r}_{\rho}) \tag{B-15}
$$

$$
+ 3\ A^{je}_{\alpha\beta}\ \tilde{a}^{r}_{\beta\gamma}\ A^{je(T)}_{\gamma\delta}\ (b^{j(T)}_{\delta k} + S^{j11(T)}_{\delta k} + S^{j13(T)}_{\delta k})\ q^{j}_{k}\}
$$

where

$$
S^{j6}_{\alpha} = d^{j}_{\alpha} + H^{j3}_{\alpha}
$$

$$
S^{j11}_{k\alpha} = a^{r(T)}_{\delta}\ A^{je(T)}_{\delta\beta}\ B^{j}_{k\alpha\beta}
$$

$$
S^{j12}_{k\ell\beta} = a^{r(T)}_{\delta}\ A^{je(T)}_{\delta\alpha}\ C^{j}_{k\ell\alpha\beta}
$$

$$
S^{j13}_{\ell\alpha} = q^{j}_{k}\ S^{j12}_{k\ell\alpha}
$$

B.3 Derivation of Aerodynamic and Solar Pressure Produced Generalized Forces

This analysis will be restricted to consideration of only a flat plate configuration. Thus, utilizing Equation (3) of Section 6.2, the general expression for the force on an element of area dA^j is given by

$$d\bar{F}^{jf} = P^f |\cos \eta^j| \{H^{jf} \cos \eta^j \; \underline{e}^{jf} + G^{jf} \; \underline{\delta}^f\} \; dA^j \tag{B-16}$$

Therefore, the virtual work done by $d\bar{F}^{jf}$ acting on an arbitrary element dA^j due to the virtual displacement

$$\delta\bar{R}^i + \delta\bar{u}^j + \delta\bar{\theta}^j \times (\bar{r}^j + \bar{u}^j)$$

is

$$\delta W^{jf} = \int_{A^j} d\bar{F}^{jf} \cdot [\delta\bar{R}^i + \delta\bar{u}^j + \delta\bar{\theta}^j \times (\bar{r}^j + \bar{u}^j)]$$

where

$$\delta\bar{R}^i = \delta R^i_\alpha \; \underline{e}^r_\alpha$$

$$\delta\bar{u}^j = \delta q^j_\ell \; \phi^j_{\ell\alpha} \; \underline{e}^j_\alpha$$

$$\delta\bar{\theta}^j = \delta\theta^j_\alpha \; \underline{e}^{jg}_\alpha \quad .$$

Also,

$$\delta W^{jf} = R^{jF}_\alpha \; \delta R^i_\alpha + Q^{jF}_k \; \delta q^j_k + QH^{jF}_\alpha \; \delta\theta^j_\alpha$$

so that

$$R^{jF}_\alpha = \int_{A^j} d\bar{F}^{jf} \cdot \underline{e}^r_\alpha$$

$$Q^{jF}_k = \int_{A^j} d\bar{F}^{jf} \cdot \underline{e}^j_\alpha \; \phi^j_{k\alpha}$$

$$Q\textcircled{H}{}^{jF}_{\alpha} = \int_{A^j} d\bar{F}^{jf} \cdot \{\underline{e}^{jg}_{\alpha} \times (\bar{r}^j + \bar{u}^j)\}$$

where R^{jF}_{α} , Q^{jF}_{k} and $Q\textcircled{H}{}^{jF}_{\alpha}$ are the generalized pressure forces associated with the generalized coordinates R^{i}_{α} , q^{j}_{k} and θ^{j}_{α} respectively.

Thus,

$$R^{jF}_{\alpha} = P^f |\cos \eta^j| \{H^{jf} \cos \eta^j \; \underline{e}^{jf} \cdot \underline{e}^{r}_{\alpha} + G^{jf} \; \underline{\delta}^f \cdot \underline{e}^{r}_{\alpha}\} A^j$$

so that

$$R^{jF}_{\alpha} = A^j \; P^f |\cos \eta^j| A^{jr(T)}_{\alpha\beta} \{H^{jf} \cos \eta^j \; B^j_{\beta} + G^{jf} \; \delta^f_{\beta}\} \tag{B-17}$$

where

$$\underline{e}^{jf} = B^j_{\beta} \; \underline{e}^j_{\beta}$$

$$\underline{\delta}^f = \delta^f_{\beta} \; \underline{e}^j_{\beta}$$

In particular then,

$$R^{jA}_{\alpha} = \rho V^2 \; A^j |\cos \eta^j| A^{jr(T)}_{\alpha\beta} \{2(1 - \sigma^j) \cos \eta^j \; B^j_{\beta} + \sigma^j [- A^{je}_{\beta\delta} \; b^r_{\delta}]\} ,$$

and with

$$\cos \eta^j = \underline{e}^{jf} \cdot \underline{\delta}^a ,$$

then

$$\cos \eta^j = B^{j(T)}_{\alpha} \; (- A^{je}_{\alpha\beta} \; b^r_{\beta}) .$$

Therefore,

$$R_{\alpha}^{jA} = - \rho V^2 \; A^j \; A_{\alpha\beta}^{jr(T)} \{2 \; (1 - \sigma^j) \; B_{\beta}^{j} \left(B_{\delta}^{j(T)} \; A_{\delta\gamma}^{je} \; b_{\gamma}^{r} \right) + \sigma^j \; A_{\beta\delta}^{je} \; b_{\delta}^{r}\} \; |B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; b_{\rho}^{r}| \; . \tag{B-18}$$

Also,

$$R_{\alpha}^{jS} = P^s \; A^j |\cos \eta^j| A_{\alpha\beta}^{jr(T)} \{2\nu^j \; \cos \eta^j \; B_{\beta}^{j} + (1 - \nu^j) \; A_{\beta\delta}^{je} \; S_{\delta}^{e}\}$$

where $\delta_{\beta}^{a} = A_{\beta\gamma}^{je} \; S_{\gamma}^{e}$, so that with $\{S^e\}_e = \{\delta^s\}_e$ (the unit sun vector), $\cos \eta^j = B_{\alpha}^{j(T)} \; A_{\alpha\beta}^{je} \; S_{\beta}^{e}$. Finally then,

$$R_{\alpha}^{jS} = P^s \; A^j \; A_{\alpha\delta}^{jr(T)} \{2\nu^j \; B_{\delta}^{j} \left(B_{\beta}^{j(T)} \; A_{\beta\gamma}^{je} \; S_{\gamma}^{e} \right) + (1 - \nu^j) \; A_{\delta\beta}^{je} \; S_{\beta}^{e}\} \; |B_{\sigma}^{j(T)} \; A_{\sigma\rho}^{je} \; S_{\rho}^{e}| \; . \tag{B-19}$$

Now looking at Q_k^{jF} ,

$$Q_k^{jF} = \int_{A^j} \phi_{k\alpha}^{j} \; e_{\alpha}^{j} \cdot P^f |\cos \eta^j| \{H^{jf} \; \cos \eta^j \; \underline{e}^{jf} + G^{jf} \; \underline{\delta}^{f}\} dA^j \; .$$

Since we are working with a flat plate, the following approximation is used:

$$\int_{A^j} \phi_{k\alpha}^{j} \; dA^j = \frac{A^j}{m^j} \int_{B^j} \phi_{k\alpha}^{j} \; dm^j$$

so that

$$\int_{A^j} \phi_{k\alpha}^{j} \, dA^j = A^j \; \phi_{k\alpha}^{j} \; .$$

Then,

$$Q_k^{jF} = A^j \quad \Phi_{k\alpha}^j \; P^f |\cos \eta^j| e_\alpha^j \cdot \{H^{jf} \cos \eta^j \; \underline{e}^{jf} + G^{jf} \; \underline{\delta}^f\}$$

and

$$Q_k^{jA} = - \rho V^2 \; A^j \; \Phi_{k\beta}^j \{2(1 - \sigma^j) \; B_\beta^j \left(B_\delta^{j(T)} \; A_{\delta\gamma}^{je} \; b_\gamma^r\right) + \sigma^j \; A_{\beta\delta}^{je} \; b_\delta^r\} |B_\sigma^{j(T)} \; A_{\sigma\rho}^{je} \; b_\rho^r| \tag{B-20}$$

and

$$Q_k^{jS} = P^s \; A^j \; \Phi_{k\delta}^j \{2\nu^j \; B_\delta^j \left(B_\beta^{j(T)} \; A_{\beta\gamma}^{je} \; S_\gamma^e\right) + (1 - \nu^j) \; A_{\delta\beta}^{je} \; S_\beta^e\} |B_\sigma^{j(T)} \; A_{\sigma\rho}^{je} \; S_\rho^e| \; . \tag{B-21}$$

Now calculating $Q \textcircled{H}_\alpha^{jF}$,

$$Q \textcircled{H}_\alpha^{jF} = \int_{A^j} \underline{e}_\alpha^{jg} \cdot (\bar{r}^j + \bar{u}^j) \times d\bar{F}^{jf}$$

$$= \underline{e}_\alpha^{jg} \cdot \int_{A^j} (\bar{r}^j + \bar{u}^j) \, dA^j \times P^f |\cos \eta^j| \{ H^{jf} \cos \eta^j \; \underline{e}^{jf} + G^{jf} \; \underline{\delta}^f\}$$

so that letting $\bar{r}^{\circ j} = \frac{1}{A^j} \int \bar{r}^j \, dA^j$ (centroid of plate) ,

$$Q \textcircled{H}_\alpha^{jF} = A^j \; \underline{e}_\alpha^{jg} \cdot (\bar{r}^{\circ j} + q_k^j \; \bar{\Phi}_{k\ell}^j) \times P^f |\cos \eta^j| \{H^{jf} \cos \eta^j \; \underline{e}^{jf} + G^{jf} \; \underline{\delta}^f\}$$

$$= A^j \; P^f \; G_{\alpha\sigma}^{j(T)} \; \{\tilde{r}_{\sigma\gamma}^{\circ j} + \tilde{H}_{\sigma\gamma}^{j3}\} \{B_\gamma^j \; H^{jf} \cos \eta^j + \delta_\gamma^f \; G^{jf}\} |\cos \eta^j| \; .$$

But, transforming from hinge axes to Body j axes,

$$ⓗ_{\alpha}^{jF} = \left(G^{j(T)}\right)_{\alpha\beta}^{-1} \; Qⓗ_{\beta}^{jF}$$

so that

$$ⓗ_{\alpha}^{jF} = A^{j} \; P^{f} \; (\tilde{r}^{\circ j}_{\alpha\gamma} + \tilde{H}^{j3}_{\alpha\gamma}) \; \{B^{j}_{\gamma} \; H^{jf} \; \cos \eta^{j} + \delta^{f}_{\gamma} \; G^{jf}\} |\cos \eta^{j}|$$

Thus,

$$ⓗ_{\alpha}^{jA} = -\rho V^{2} \; A^{j} \; (\tilde{r}^{\circ j}_{\alpha\beta} + \tilde{H}^{j3}_{\alpha\beta}) \; \{2(1 - \sigma^{j}) \; B^{j}_{\beta} \left(B^{j(T)}_{\delta} \; A^{je}_{\delta\gamma} \; b^{r}_{\gamma} \right) + \sigma^{j} \; A^{je}_{\beta\delta} \; b^{r}_{\delta}\} |B^{j(T)}_{\sigma} \; A^{je}_{\sigma\rho} \; b^{r}_{\rho}| \tag{B-22}$$

and

$$ⓗ_{\alpha}^{jS} = P^{s} \; A^{j} \; (\tilde{r}^{\circ j}_{\alpha\beta} + \tilde{H}^{j3}_{\alpha\beta}) \{2 \nu^{j} \; B^{j}_{\beta} \left(B^{j(T)}_{\delta} \; A^{je}_{\delta\gamma} \; b^{r}_{\gamma} \right) + (1 - \nu^{j}) \; A^{je}_{\beta\delta} \; S^{e}_{\delta}\} |B^{j(T)}_{\sigma} \; A^{je}_{\sigma\rho} \; b^{r}_{\rho}| \tag{B-23}$$

Appendix C. Modification of UFSSP to Simulate The Dynamics of Rapidly Spinning Flexible Systems

The material for this appendix has been taken from Reference 2 which contains a more complete description of the modification including test cases. The equations contained herein are presented in their original form with the important exception that only first order terms have been coded so that only the first order correction terms are presented in Equation (4.4).

1.0 INTRODUCTION

The modification described herein extends the capabilities of the program to provide for the accurate simulation of rapidly spinning flexible spacecraft, flexible spacecraft having variable angular rates, and the structural dynamics of helicopter rotors. In general, the modification provides the capability of simulating the dynamics of rapidly spinning systems of bodies or systems with variable spin in the configuration of a topological tree having terminal flexible bodies. To use this special option of UFSSP, the terminal flexible bodies must be representible as space curves, however, the program will predict the general bending and torsional motion of the body including centrifugal stiffening effects.

When a beam--a spacecraft appendage or helicopter rotor for example, rotates about an axis perpendicular to its own axis, the resulting centrifugal forces have the effect of stiffening the blade to a certain degree thereby increasing the natural frequency of bending vibration of the beam. Traditionally, this effect has been accounted for by deriving a centrifugal potential energy term which is added to the elastic energy of the system or by summing the forces on portions of the beam mass. The present approach differs in that the appropriate terms are derived from a modified displacement function for the flexible body. That is, the original displacement function for the body $\bar{u}^*$ is replaced by

$$\bar{u} = \bar{u}^* - \frac{1}{2} \int_0^S \left(\frac{d\bar{u}^*}{d\eta}\right) \cdot \left(\frac{d\bar{u}^*}{d\eta}\right) \frac{d\bar{r}(\eta)}{d\eta} \, d\eta \tag{1.1}$$

where $\bar{r}$ is the position vector of a point on the flexible space curve.

When used in the derivation of the kinetic energy, Equation (1.1), yields the approprate terms to account for centrifugal stiffening as well as for the effects of other components of acceleration.

In the next section, Section 2, the geometry defining the terminal flexible space curves and their deflections is considered. In the following section, Section 3, the modifications to UFSSP equations of motion are derived. These results are summarized in Section 4 and the method of their incorporation in UFSSP is outlined.

2.0 GEOMETRY ASSOCIATED WITH DEFORMED SPACE CURVES

In the present version of UFSSP, flexibility is modeled as a summation of separable functions of displacement and time. That is, the displacement of an arbitrary point $\bar{r}^j$ in Body j at time t is given by

$$\bar{u}^{*j} \ (\bar{r}^j, t) = \sum_{i=1}^{n_j} q_i^j(t) \ \bar{\phi}_i^j \ (\bar{r}^j) \tag{2.1}$$

The equations of motion as they presently exist include all terms which arise from spin and other components of system motion on the first order displacement, $\bar{\phi}_i^j$ $(\bar{r})$. Therefore, any additional terms to be included in the equations of motion must arise from higher order displacements Flexural shortening is the second order displacement of Body j and therefore logically must account for the additional terms required. This term arises because the displacement functions, $\bar{\phi}_i^j$, are defined over the entire surface of Body j and the distance along the deformed curve to a particular point is greater than the distance along the same path on the undeformed body. Consequently to insure inextensibility, the deformed shape $\bar{u}^{*j}$ $(\bar{r}^i, t)$ must be corrected by an additional term.

The correction term which is derived here is applicable to flexible bodies in the configuration of a spacecurve, that is, one dimensional flexible bodies. More complex geometries, such as a closed loop or a surface, present additional difficulties in that the net distance traversed around every closed path on the deflected body must be zero. This is a solvable problem whose solution mainly depends on the resolution of indexing problems and the establishment of suitable numerical algorithms. However, it is more complex in the case of structures of higher dimension than for space curves. However, restricting the modification to space curves yields results useful to many practical problems.

Figure 2.1 shows Body j, a flexible space curve, with its undeformed shape defined by the vector function $\bar{r}(s)$, where s is the arc length along the undeformed body measured from the origin. The uncorrected displaced form of the body is defined by the vector function $\bar{d}(s,t)$ measured from the origin (a fixed point) along the deflected structure. The vectors $\bar{r}$ and $\bar{d}$ are related as follows:

$$\bar{d}\,(s) = \bar{r}\,(s) + \bar{u}^*\,(s,t) \tag{2.2}$$

where $\bar{u}^*\,(s,t)$ is defined in Equation (2.1) and the superscript has been dropped since it is unimportant to the discussion of this Section to distinguish Body j from other bodies.

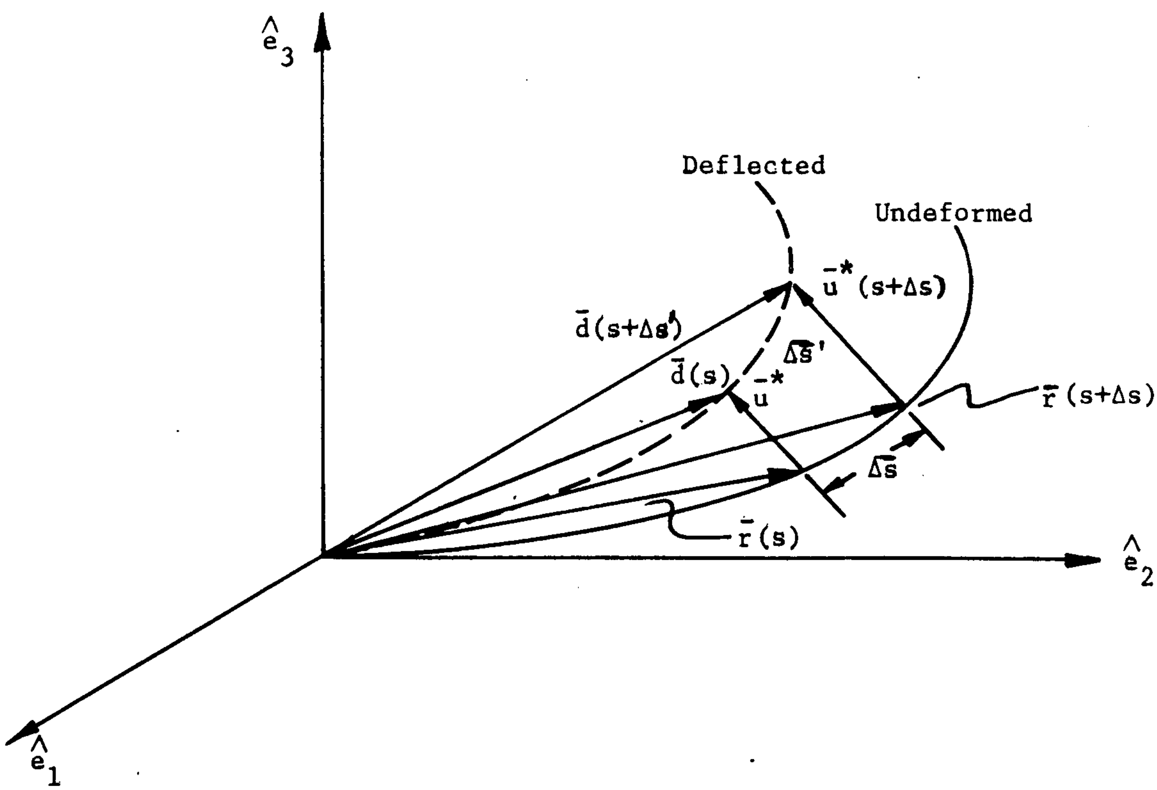


Figure 2.1. Body j, A Flexible Space Curve

A small distance, $\bar{\Delta s}'$, along the deflected Body j the position vector is

$$\bar{d}(s + \Delta s') = \bar{r}(s + \Delta s) + \bar{u}^*(s + \Delta s, t) \tag{2.3}$$

An element of arc on the deformed curve is then given by

$$\bar{\Delta s}' = \bar{d}(s + \Delta s') - \bar{d}(s) \tag{2.4}$$

or

$$\bar{\Delta s}' = \bar{r}(s + \Delta s) - \bar{r}(s) + \bar{u}^*(s + \Delta s, t) - \bar{u}^*(s,t) \tag{2.5}$$

and, in the limit

$$ds' = \frac{d\bar{r}(s)}{ds}\, ds + \frac{d\bar{u}^*(s,t)}{ds}\, ds \tag{2.6}$$

The length of the arc differential $d\bar{s}'$ is

$$ds' = \left(\sqrt{\frac{d\bar{r}}{ds} \cdot \frac{d\bar{r}}{ds} + 2\,\frac{d\bar{r}}{ds} \cdot \frac{d\bar{u}^*}{ds} + \frac{d\bar{u}^*}{ds} \cdot \frac{d\bar{u}^*}{ds}}\right) ds \tag{2.7}$$

where it is to be noted that

$$\left|\frac{d\bar{r}}{ds}\right| = 1 \tag{2.8}$$

Since $\frac{d\bar{r}}{ds}$ is the unit vector tangent to the space curve at s and

$$\frac{d\bar{r}}{ds} \cdot \frac{d\bar{u}^*}{ds} = 0\,, \tag{2.9}$$

that is, the incremental displacement is normal to the corresponding incremental length of the structure.

Using Equations (2.8) and (2.9), Equation (2.7) becomes

$$ds' = \sqrt{1 + \frac{d\bar{u}^*}{ds} \cdot \frac{d\bar{u}^*}{ds}} \; ds \tag{2.10}$$

The quantity $\frac{d\bar{u}}{ds} \cdot \frac{d\bar{u}}{ds}$ <<1, therefore we use the approximation

$$ds' = \left(1 + \frac{1}{2} \frac{d\bar{u}}{ds} \cdot \frac{d\bar{u}}{ds}\right) ds \tag{2.11}$$

The differential flexural shortening is given by the following vector

$$(ds - ds') \frac{d\bar{r}(s)}{ds} = \left(- \frac{1}{2} \frac{d\bar{u}}{ds} \cdot \frac{d\bar{u}}{ds} \; ds\right) \frac{d\bar{r}(s)}{ds} \tag{2.12}$$

The original displacement function for the body, $\bar{u}^*(s)$, may be replaced by the following function

$$\bar{u}(s,t) = \bar{u}^*(s,t) + \bar{u}^m(s,t) \tag{2.13}$$

where $\quad \bar{u}^m(s,t) = - \frac{1}{2} \int_0^s \frac{d\bar{u}^*}{d\eta} \cdot \frac{d\bar{u}^*}{d\eta} \; \frac{d\bar{r}(\eta)}{d\eta} \; d\eta$

and η is a dummy arc length variable.

Substitution of Equation (2.1) into the second term of Equation (2.13) yields

$$\bar{u}^m = \sum_{i=1}^{n} \sum_{k=1}^{n} q_i(t) \, q_k(t) \, \bar{F}_{ik} \tag{2.14}$$

where $\quad \bar{F}_{ik} = - \frac{1}{2} \int_0^s \left(\frac{d\bar{\phi}i}{d\eta} \cdot \frac{d\bar{\phi}k}{d\eta}\right) \frac{d\bar{r}}{d\eta} \; d\eta$

3.0 DERIVATION OF THE MODIFICATIONS REQUIRED TO THE UFSSP EQUATIONS OF MOTION

As stated in Equation (2.13) the displacement function for the flexible body, $\bar{u}^{*}(s,t)$ can be corrected for flexural shortening by adding a term $\bar{u}^{mj}$ (s,t) to $\bar{u}^{*}$ (s,t). The corrected displacement function is

$$\bar{u}^{j}\ (s,t) = \bar{u}^{*j}(s,t) + \bar{u}^{mj}(s,t) \tag{3.1}$$

where $$\bar{u}^{mj}(s,t) = \sum_{i=1}^{n} \sum_{k=1}^{n} q_i^j\, q_k^j\, \bar{F}_{ik}^j$$

This modified expression for $\bar{u}^{j}$ may now be used to modify the expression for the position of an arbitrary mass point in Body j, $\bar{\rho}^{j}$, and subsequently, the kinetic energy expression for Body j.

The instantaneous position vector $\bar{\rho}^{j}$ to an arbitrary point in Body j may now be written

$$\bar{\rho}^{j} = \bar{R}^{r} + \bar{R}^{i} + \bar{\ell}_{ij} + \bar{r}^{j} + \bar{u}^{j} \tag{3.2}$$

where $$\bar{u}^{j} = \bar{u}^{*j} + \bar{u}^{mj}$$

The velocity vector of an arbitrary field point in the flexible Body j is found from Equation (3.2) to be

$$\frac{d\bar{\rho}^{j}}{dt} = \dot{\bar{R}}^{r} + \dot{\bar{R}}^{i} + \bar{\omega}^{r} \times \bar{R}^{i} + \dot{\bar{\ell}}^{ij} + \bar{\omega}^{i} \times \bar{\ell}^{ij}$$

$$+ \bar{\omega}^{j} \times \bar{r}^{j} + \dot{\bar{u}}^{j} + \bar{\omega}^{j} \times \bar{u}^{j} \tag{3.3}$$

Some terms in Equation (3.3) are constant over Body j, it will simplify the process of evaluation of the kinetic energy integral to group these in one term. Redefine $\frac{d\bar{\rho}^j}{dt}$ as

$$\frac{d\bar{\rho}^j}{dt} = \bar{x}^{j1} + \bar{\omega}^j \times \bar{r}^j + \dot{\bar{u}}^j + \bar{\omega}^j \times \bar{u}^j$$

where $$\bar{x}^{j1} = \dot{\bar{R}}^r + \dot{\bar{R}}^i + \bar{\omega}^r \times \bar{R}^i + \dot{\bar{\ell}}^{ij} + \bar{\omega}^i \times \bar{\ell}^{ij} \tag{3.4}$$

Substituting the expression for $\bar{u}^j$, Equation (3.1), into Equation (3.3) we obtain

$$\frac{d\bar{\rho}^j}{dt} = \frac{d\rho}{dt}^{*j} + \dot{\bar{u}}^{mj} + \bar{\omega}^j \times \bar{u}^{mj} \tag{3.5}$$

where

$$\frac{d\rho}{dt}^{*j} = \bar{x}^{j1} + \bar{\omega}^j \times \bar{r}^j + \dot{\bar{u}}^{*j} + \bar{\omega}^j \times \bar{u}^{*j} ,$$

the previous definition of $\frac{d\bar{\rho}^j}{dt}$.

The square of the velocity of an arbitrary point in Body j is then

$$\frac{d\bar{\rho}^j}{dt} \cdot \frac{d\bar{\rho}^j}{dt} = \frac{d\bar{\rho}^{*j}}{dt} \cdot \frac{d\bar{\rho}^{*j}}{dt} + 2 \frac{d\bar{\rho}^{*j}}{dt} \cdot \dot{\bar{u}}^{mj}$$

$$+ 2 \frac{d\bar{\rho}^{*j}}{dt} \cdot (\bar{\omega}^j \times \bar{u}^{mj}) + \dot{\bar{u}}^{mj} \cdot \dot{\bar{u}}^{mj}$$

$$+ 2 \dot{\bar{u}}^{mj} \cdot (\bar{\omega}^j \times \bar{u}^{mj}) + (\bar{\omega}^j \times \bar{u}^{mj}) \cdot (\bar{\omega}^j \times \bar{u}^{mj})$$

The last three terms in Equation (3.6) are fourth order in q , $\dot{q}$ and will thus lead to third order terms in the equations of motion. Consequently, these terms may be neglected from this point. Using Equation (3.6), the kinetic energy of Body j may be written as follows:

$$T = \frac{1}{2} \int_{\beta_j} \frac{d\bar{\rho}^j}{dt} \cdot \frac{d\bar{\rho}^j}{dt} \, dm^j$$

$$= T^* + \int_{\beta_j} \frac{d\bar{\rho}^{*j}}{dt} \cdot \dot{\bar{u}}^{mj} \, dm^j + \int_{\beta_j} \frac{d\bar{\rho}^*}{dt} \cdot (\bar{\omega}^j \times \bar{u}^{mj}) \, dm^j \tag{3.7}$$

Substituting the second of Equations (3.5) into Equation (3.7), we obtain an expression for the kinetic energy which is the sum of eight integrals:

$$\begin{aligned} T = {} & T^* + \bar{x}^{j1} \cdot \overset{t1}{\int_{\beta_j}} \dot{\bar{u}}^{mj} \, dm^j + \bar{x}^{j1} \cdot \left(\bar{\omega}^j \times \overset{t2}{\int_{\beta_j}} \bar{u}^{mj} \, dm^j \right) \\ & + \int_{\beta_j} \overset{t3}{(\bar{\omega}^j \times \bar{r}^j) \cdot \dot{\bar{u}}^{mj}} \, dm^j + \int_{\beta_j} \overset{t4}{\dot{\bar{u}}^{*j} \cdot \dot{\bar{u}}^{mj}} \, dm^j \\ & + \int_{\beta_j} \overset{t5}{(\bar{\omega}^j \times \bar{u}^{*j}) \cdot \dot{\bar{u}}^{mj}} \, dm^j + \int_{\beta_j} \overset{t6}{(\bar{\omega}^j \times \bar{r}^j) \cdot (\bar{\omega}^j \times \bar{u}^{mj})} \, dm^j \\ & + \int_{\beta_j} \overset{t7}{\dot{\bar{u}}^{*j} \cdot (\bar{\omega}^j \times \bar{u}^{mj})} \, dm^j + \int_{\beta_j} \overset{t8}{(\bar{\omega}^j \times \bar{u}^{*j}) \cdot (\bar{\omega}^j \times \bar{u}^{mj})} \, dm^j \end{aligned} \tag{3.8}$$

where T^* is the previous expression for the kinetic energy.

From Equation (3.1) we have

$$\bar{u}^{mj} = \sum_{i,k} q_i^j \, q_k^j \, \bar{F}_{ik}^j \tag{3.9}$$

Consequently,

$$\int_{\beta_j} \bar{u}^{mj}\, dm^j = m^j \sum_{i,k} q_i^j\, q_k^j\, \bar{F}^j_{(1)ik} \tag{3.10}$$

where $\bar{F}^j_{(1)ik} = \dfrac{1}{m^j} \displaystyle\int \bar{F}^j_{ik}\, dm^j$ is a new special mass property. This quantity may also be used to define the integral

$$\int_{\beta_j} \dot{\bar{u}}^{mj}\, dm^j = 2\, m^j \sum_{i,k} q_i^j\, \dot{q}_k^j\, \bar{F}^j_{(1)ik} \tag{3.11}$$

We may now evaluate the kinetic energy term by term.

$$t1 = 2\, m^j\, \dot{\bar{x}}^{j1} \cdot \sum_{i,k} q_i^j\, \dot{q}_k^j\, \bar{F}^j_{(1)ik} \tag{3.12}$$

$$t2 = m^j\, \dot{\bar{x}}^{j1} \cdot \left(\bar{\omega}^j \times \sum_{i,k} q_i^j\, q_k^j\, \bar{F}^j_{(1)ik}\right) \tag{3.13}$$

$$t3 = \int_{\beta_j} (\bar{\omega}^j \times \bar{r}^j) \cdot \dot{\bar{u}}^{mj}\, dm^j$$

$$= m^j\, \bar{\omega}^j \cdot \sum_{i,k} q_i^j\, q_k^j\, \bar{Z}^{mj}_{ik} \tag{3.14}$$

where $\bar{Z}^{mj}_{ik} = \dfrac{2}{m^j} \displaystyle\int_{\beta_j} \bar{r}^j \times \bar{F}^j_{ik}\, dm^j$

The terms t4 and t5 will result in second order contributions to the equations of motion, consequently these are neglected.

$$t6 = \int_{\beta_j} (\bar{\omega}^j \times \bar{r}^j) \cdot (\bar{\omega}^j \times \bar{u}^{mj})\, dm^j$$

$$= \frac{m^j}{2}\, \bar{\omega}^j \cdot \sum_{i,k} q_i^j\, q_k^j\, \bar{\bar{E}}^{mj}_{ik} \cdot \bar{\omega}^j \tag{3.15}$$

where

$$\bar{\bar{E}}^{mj}_{k\ell} = \frac{2}{m^j} \int_{\beta_j} \left((\bar{r}^j \cdot \bar{F}^j_{k\ell})\bar{\bar{\delta}} - \bar{r}^j \, \bar{F}^j_{k\ell} \right) dm^j$$

and $\bar{\bar{\delta}}$ is the Kronecker delta.

The remaining terms t7 and t8 are second order.

The previous results may be combined to write the kinetic energy for Body j as follows:

$$T = T_1 + T_2 \tag{3.16}$$

where

$$T_1 = T^* + \bar{\omega}^j \cdot \left(m^j \sum_{i,k} q^j_i \, q^j_k \, \bar{Z}^{mj}_{i\,k} \right)$$

$$+ \frac{m^j}{2} \bar{\omega}^j \cdot \left(\sum_{i,k} q^j_i \, q^j_k \, \bar{\bar{E}}^{mj}_{ik} \right) \cdot \bar{\omega}^j$$

$$T_2 = m^j \, \bar{x}^{j1} \cdot \left\{ 2 \sum_{i,k} q^j_i \, \dot{q}^j_k \, \bar{F}^j_{(1)ik} \right.$$

$$\left. + \bar{\omega}^j \times \sum_{i,k} q^j_i \, q^j_k \, \bar{F}^j_{(1)ik} \right\}$$

The terms in T_1 are of the same form as certain terms in T^*, that is, in T^* we have the terms

$$\frac{m^j}{2} \bar{\omega}^j \cdot \left(\sum_{i,k} q^j_i \, q^j_k \, \bar{\bar{E}}^j_{ik} \right) \cdot \bar{\omega}^j + \bar{\omega}^j \cdot \left(m^j \sum_{i,k} q^j_i \, \dot{q}^j_k \, \bar{Z}^j_{ik} \right) \tag{3.17}$$

Thus, the terms in T_1 can be accounted for in the original equations of motion by redefining $\bar{\bar{E}}^j_{ik}$ and $\bar{Z}^j_{ik}$ as follows:

$$\bar{\bar{E}}^{j}_{ik} = \bar{\bar{E}}^{*j}_{ik} + \bar{\bar{E}}^{mj}_{ik}$$

$$\bar{Z}^{j}_{ik} = \bar{Z}^{*j}_{ik} + \bar{Z}^{mj}_{ik} \tag{3.18}$$

where $\bar{\bar{E}}^{*j}_{ik}$ and $\bar{Z}^{*j}_{ik}$ replace $\bar{\bar{E}}^{j}_{ik}$ and $\bar{Z}^{j}_{ik}$ as they are originally defined and $\bar{\bar{E}}^{mj}_{ik}$, and $\bar{Z}^{mj}_{ik}$, are modifications that must be added to these to correct for flexural shortening.

The terms contributed by T_2 to the equations of motion are established by means of Lagrange's Equations.

4.0 SUMMARY OF FORMULATION RESULTS AND APPROACH TO INCORPORATION IN UFSSP

4.1 Summary of the Equations of Motion

The modified equations of motion are as follows:

The q^i_k Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i_k}\right) - \frac{\partial T}{\partial q^i_k} = \frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{q}^i_k}\right) - \frac{\partial T_1}{\partial q^i_k}$$

$$+ \Big(2\, m^j\, F^{j2}_{k\alpha}\, A^{jr}_{\alpha\gamma}\, \ddot{R}^i_{\gamma} - 2\, m^j\, F^{j2}_{k\alpha}\, A^{ji}_{\alpha\gamma}\, \tilde{\ell}^{ij}_{\gamma\sigma}\, \dot{\omega}^i_{\sigma}$$

$$+ 2\, m^j\, F^{j2}_{k\alpha}\, (A^{je}_{\alpha\gamma}\, \ddot{R}^r_{\gamma} + S^{j7}_{\alpha})\Big) \tag{4.1}$$

where $f^{j2}_{k\alpha} = q^j_e\, F^j_{(1)\ell k\alpha}$

The R^i_α Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}^i_\alpha}\right) - \frac{\partial T}{\partial R^i_\alpha} = \frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{R}^i_\alpha}\right) - \frac{\partial T}{\partial R^i_\alpha}$$

$$+ \int m^j \, A^{jr(T)}_{\alpha\beta} \Big[F^{j3}_\beta + 2 \, \bar{F}^{j2(T)}_{\beta k} \, \ddot{q}^j_k - \tilde{F}^{j1}_{\beta\gamma} \, \dot{\omega}^j_\gamma \tag{4.2}$$

$$+ 4 \, \tilde{\omega}_{\beta\gamma} \, F^{j5}_\gamma + \tilde{\omega}^j_{\beta\gamma} \, \tilde{\omega}^j_{\beta\rho} \, F^{j1}_\rho \Big]$$

where $F^{j1}_\alpha = q^j_k \, F^{j2}_{k\alpha}$

$$F^{j3}_\alpha = 2 \, \dot{q}^j_\ell \, \dot{q}^j \, \dot{q}^j_k \, F^j_{(1)\ell k\alpha}$$

$$F^{j6}_\alpha = \dot{q}^j_k \, F^{j2}_{k\alpha}$$

The θ^i_λ Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}^i_\lambda}\right) - \frac{\partial T}{\partial \theta^i_\lambda} = \frac{d}{dt}\left(\frac{\partial T_1}{\partial \theta^i_\lambda}\right) - \frac{\partial T_1}{\partial \theta^i_\lambda}$$

$$+ m^j \, \tilde{F}^{j1}_{\lambda\alpha} \, (A^{je}_{\alpha\gamma} \, \ddot{R}^r_\gamma + A^{jr}_{\alpha\gamma} \, \ddot{R}^i_\gamma - A^{ji}_{\alpha\gamma} \, \tilde{\ell}^{ij}_{\gamma\sigma} \, \dot{\omega}^i_\sigma + S^{j7}_\alpha) \tag{4.3}$$

The above relations will now be used to redefine the elements of the B matrix and C matrix defined by Equations (5-35) to (5-49). Old expressions for the sub-matrices will be designated by a superscript*. (Note: only first-order correction terms are included here.)

$$B^{A11}_{k\ell} = B^{A11^*}_{k\ell}$$

$$B^{A12}_{k\gamma} = B^{A12^*}_{k\gamma}$$

$$B^{A13}_{k\beta} = B^{A13^*}_{k\beta} - 2\ m^j \sum_{\alpha=1}^{3} F^{j2}_{k\alpha}\ A^{ji}_{\alpha\gamma}\ \tilde{\ell}^{ij}_{\gamma\beta}$$

$$B^{A14}_{k\beta} = B^{A14^*}_{k\beta} + 2\ m^j \sum_{\alpha=1}^{3} F^{j2}_{k\alpha}\ A^{jr}_{\alpha\beta}$$

$$C^{A1}_{k} = C^{A1^*}_{k} - 2\ m^j\ F^{j2}_{k\alpha}\ S^{j7}_{\alpha}$$

$$B^{A21}_{\delta\ell} = B^{A21^*}_{\delta\ell}$$

$$B^{A22}_{\delta\gamma} = B^{A22^*}_{\delta\gamma}$$

$$B^{A23}_{\delta\beta} = B^{A23^*}_{\delta\beta}$$

$$B^{A24}_{\delta\beta} = B^{A24^*}_{\delta\beta}$$

$$C^{A2}_{\delta} = C^{A2^*}_{\delta}$$

$$B^{A31}_{\alpha\ell} = B^{A31^*}_{\alpha\ell} + \sum_{\beta=1}^{3} 2\, m^j \; A^{jr(T)}_{\alpha\beta} \; F^{j2(T)}_{\beta\ell}$$

$$B^{A32}_{\alpha\gamma} = B^{A32^*}_{\alpha\gamma}$$

$$B^{A33}_{\alpha\beta} = B^{A33^*}_{\alpha\beta}$$

$$B^{A34}_{\alpha\beta} = B^{A34^*}_{\alpha\beta}$$

$$C^{A3}_{\alpha} = C^{A3^*}_{\alpha} \qquad (4.4)$$

Various quantities used in Equation (4.4) are auxiliary quantities which are calculated at each time step. These quantities are generally expressible in terms of the special mass properties. In the following, the calculation of the special mass properties is clarified.

4.2 New Special Mass Properties Required

The UFSS Program has been modified to include additional terms for a terminal body which is flexible and in the configuration of a space curve. Any space curve may be approximated by a sequence of interconnected points, and consequently, the most logical representation of a space curve is in terms of a series of interconnected nodal points. All of the modifications to the special mass properties are derived in terms of such a model. However, it may be desired to model a structure having significant lateral dimensions as shown in Figure 4.1 which requires several rows of nodes.

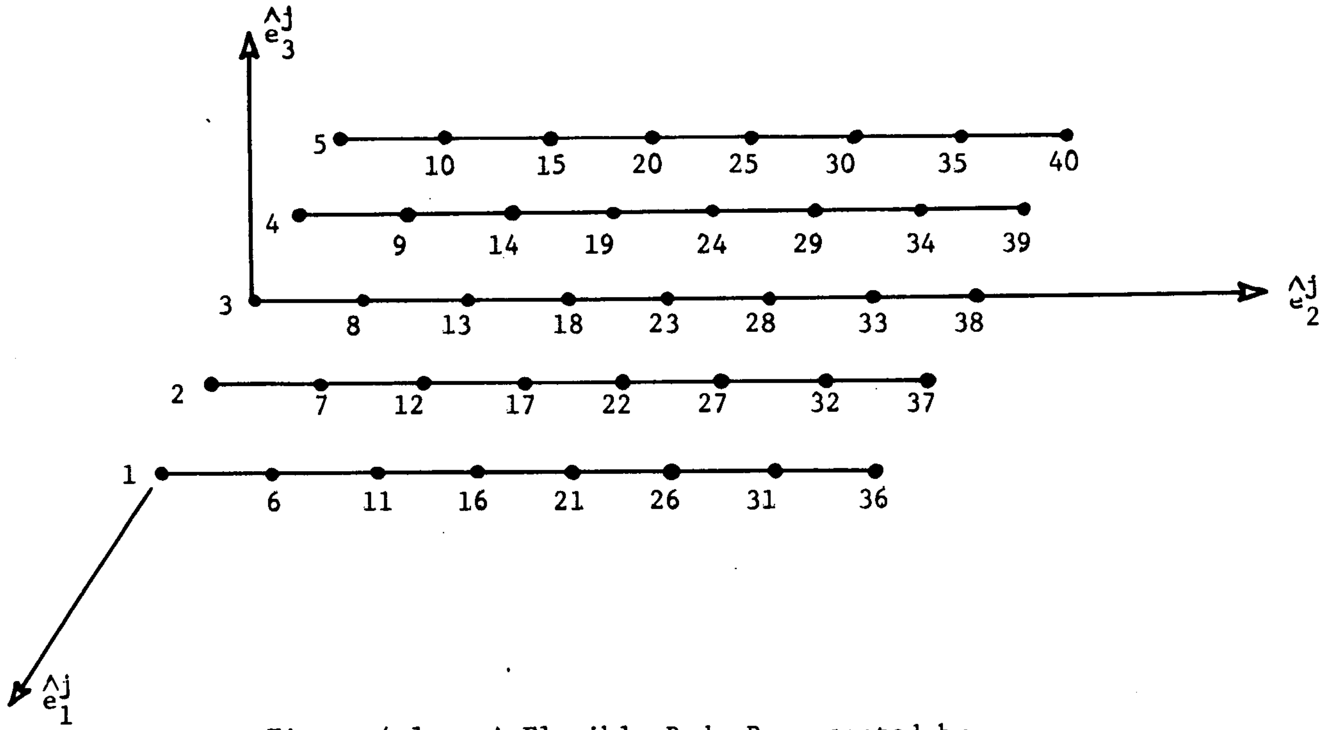


Figure 4.1. A Flexible Body Represented by Several Rows of Nodes

The quantities associated with the rows of nodes defining a structure such as the structure of Figure 4.1 must be used to define the properties of the system represented by a single row of nodes as shown in Figure 4.2

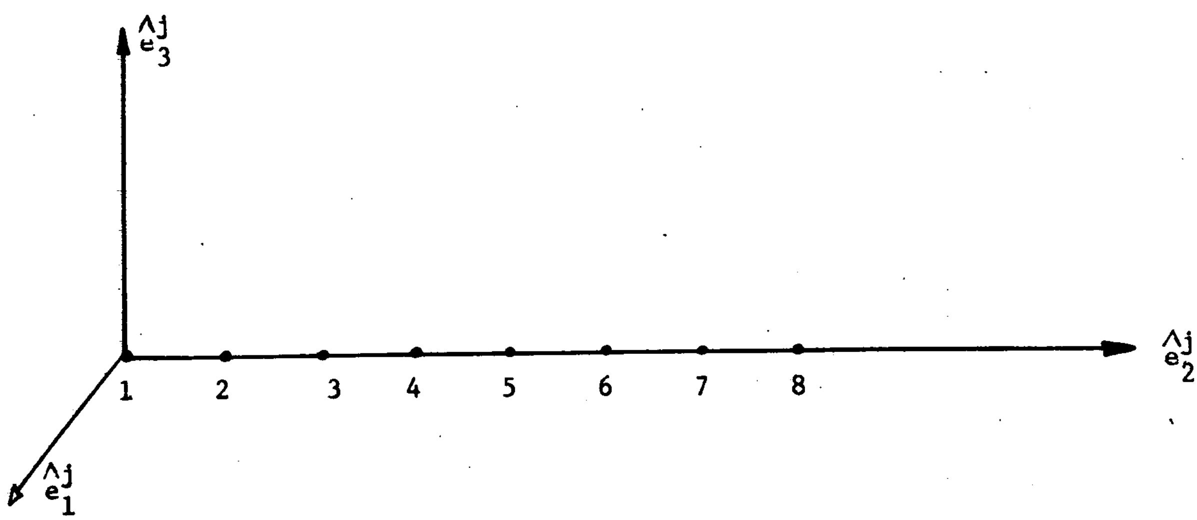


Figure 4.2. Nodal Numbering Scheme Admissable to the Analysis

The quantities associated with the structure of Figure 4.1 will be designated as follows:

Δm_{ℓ}^{*j} = the mass lumped at the ℓth node of Body j

$x_{\alpha\ell}^{*j}$ = the αth coordinate (α=1,2,3) of node ℓ in Body j

$\bar{\phi}_{\alpha\ell k}^{*j}$ = the αth component of mode k at node ℓ

m_j^* = the total number of nodes ($m_j^* \leq 64$)

These quantities are all input to the program in its original form, but the user is now constrained to number the nodes as shown in Figure 4.1. These quantities must be used to obtain equivalent properties for a space curve as shown in Figure 4.2. One additional input is required, the number of rows of nodes, however this can be accounted for by reinterpretation of the significance of the input flag designating whether a body is rigid or flexible. Thus, let f_j equal the number of rows of nodes, then the mass lumped at the i^{th} node in the space curve is given by

$$\Delta m_i^j = \sum_{\ell = f_j(i-1)+1}^{f_j i} \Delta m_{\ell}^{*j} \quad ; \quad i = 1,\ldots,m_j$$

$$\text{where} \quad m_j = m_j^*/f_j \tag{4.5}$$

Thus, for example, for the structure shown in Figure 4.2 we obtain the mass at the third node from the nodal masses of the structure in Figure 4.1 as follows:

$$i = 3$$

$$f_j = 5$$

$$\Delta m_3^j = \sum_{\ell = 5(3-1)+1}^{5(3)} \Delta m_\ell^{*j}$$

$$= \sum_{\ell=11}^{15} \Delta m_\ell^{*j}$$

The coordinates of the nodes and the components of the modes of the representation shown in Figure 4.2 will be assumed to be those associated with the center row of nodes in the structure of Figure 4.1. Consequently, f_j must always be an odd number. The number of the node in the representation of Figure 4.1, node ℓ , which corresponds to node i in the structure of Figure 4.2 is found from

$$\ell = \frac{1}{2} (2 f_j i - f_j + 1) \qquad (4.6)$$

For example, for $f_j = 5$, and $i = 3$ we obtain

$$\ell = \frac{1}{2} (2(5)3 - 5 + 1) = 13$$

The modes and coordinates of at node i in the representation shown in Figure 4.2 are found as follows:

$$\phi_{\alpha ik}^{j} = \phi_{\alpha \ell k}^{*j} \qquad (4.7)$$

where $\ell = \frac{1}{2} (2f_j i - f_j + 1)$ and

$$x_{\alpha i}^{j} = x_{\alpha \ell}^{*j} \qquad (4.8)$$

Preliminary to defining the modifications to the special mass properties, the following quantities will be required, none of which are output.

$$\Delta r^j_{\alpha i} = x^j_{\alpha i} - x^j_{\alpha,i-1} \qquad \begin{array}{l} \alpha = 1,2,3 \\ i = 2,3,\ldots m_j \end{array}$$

$$\Delta r^j_{\alpha 1} = x^j_{\alpha 1}$$

$$L^j_i = \left(\sum_{\alpha=1}^{3} (\Delta r^j_{\alpha i})^2 \right)^{1/2}$$

and a vector is defined at each node, h , for modes k and ℓ ,

$$F^j_{\alpha k \ell h} =$$

$$- \frac{1}{2} \sum_{i=1}^{h} \Delta r_{\alpha \ell} \frac{1}{(L^j_i)^2} \sum_{\beta=1}^{3} (\phi^j_{\beta k i} - \phi^j_{\beta k,k-1}) (\phi^j_{\beta \ell i} - \phi^j_{\beta \ell, i-1}) \tag{4.9}$$

Component β

mode k

node i $\qquad h = 1, 2 \ldots , m_j$

The following quantities will be calculated. The first, $\bar{F}^j_{(1)k\ell}$ is a special mass property which arises in base motion terms, while the second and third are terms which add to the existing special mass property terms. These additional special mass properties can be expressed in terms of integrals which take the same form as other existing special mass property integrals.

The special mass property $\bar{F}^j_{(1)k\ell}$ is defined as follows:

$$\bar{F}^j_{(1)k\ell} = \frac{1}{m^j} \int_{B_j} \bar{F}^j_{k\ell} \, dm^j \tag{4.10}$$

This integral is replaced by a summation over the configuration of Figure 4.2, specifically by a summation over the ℓ nodes defined by (4.6).

Except for the dual subscripting, the special mass property, $\bar{F}_{(1)k\ell}$, defined in (4.10) is in exactly the same form as the integral defining $\bar{\phi}^j_\ell$ in Equation (A-3).

The second special mass property to be defined, $\bar{Z}^{mj}_{k\ell}$, is a modification to be added to an existing special mass property, $\bar{Z}^j_{k\ell}$. $\bar{Z}^{mj}_{k\ell}$ is defined as follows:

$$\bar{Z}^{mj}_{k\ell} = \frac{2}{m^j} \int_{B_j} \bar{r}^j \times \bar{F}^j_{k\ell} \; dm^j \tag{4.11}$$

This expression, except for the dual subscripting is in exactly the same form as the expression for the existing special mass property $\bar{Y}^j$ defined in Equation (A-4).

The remaining special mass property, $\bar{\bar{E}}^{mj}_{k\ell}$, is a modification to be added to an existing special mass property, $\bar{\bar{E}}^j_{k\ell}$. $\bar{\bar{E}}^{mj}_{k\ell}$ is defined as follows:

$$\bar{\bar{E}}^{mj}_{k\ell} = \frac{2}{m^j} \int_{B_j} \left(\bar{\bar{\delta}} \, (\bar{r}^j \cdot \bar{F}^j_{k\ell}) - \bar{r}^j \; \bar{F}^j_{k\ell} \right) dm^j \tag{4.12}$$

This expression, except for dual subscripting, is in exactly the same form as the expression for the existing special mass property $\bar{\bar{N}}^j_\ell$ defined in Equation (A-7).

Appendix D. Modification of UFSSP to Calculate Dynamic Loads

This appendix describes the methodology involved in calculating dynamic loads within the UFSS program. Determination of these loads can be divided into two basic phases. In the first phase, forces and moments acting at each interconnection between adjacent bodies are calculated within the UFSS program via information available from the original dynamics subroutines. In the second phase, the mode-acceleration method is utilized to calculate internal loads at any desired node point within a given terminal flexible body. These internal loads are calculated by a separate, stand-alone program operating upon a special loads history tape generated by the UFSS program.

D.1 Interconnection Force and Torque Calculations

The synthesizing algorithm within UFSSP is based on elimination of the interaction forces and torques between adjacent bodies of the system model. By so doing, the final system of matrix differential equations includes only the unconstrained degrees of freedom of the spacecraft model. This retention of only the minimal number of degrees of freedom contributes greatly to reducing the cost of the solution for the dynamic response. However, when the dynamic loads are desired, the interaction forces and torques must be obtained explicitly. Specifically, they are calculated from the linear and angular velocities and accelerations and the coordinate transformations associated with the individual bodies of the spacecraft model by following the same sequencing algorithm used to synthesize the dynamic equations of motion.

D.1.1 Interconnection Loads for a Flexible Body

If Body j is a flexible body (with limb Body i), its interconnection forces and torques can readily be determined using the equations from Appendix A. Specifically, from Equation (A-9):

$$F_{\alpha}^{ji} = A_{\alpha\beta}^{jr} \left\{ -C_{\beta}^{A3} + B_{\beta\ell}^{A31} \ddot{q}_{\ell} + B_{\beta\gamma}^{A32} \ddot{\theta}_{\gamma}^{j} + B_{\beta\delta}^{A33} \dot{\omega}_{\delta}^{i} + B_{\beta\delta}^{A34} \ddot{R}_{\delta}^{i} \right\} . \tag{D-1}$$

Similarly, from Equation (A-12):

$$T_{\alpha}^{ji} = -C_{\alpha}^{A2} + B_{\alpha\ell}^{A21}\ \ddot{q}_{\ell}^{j} + B_{\alpha\gamma}^{A22}\ \ddot{\theta}_{\gamma}^{j} + B_{\alpha\beta}^{A23}\ \dot{\omega}_{\beta}^{i} + B_{\alpha\beta}^{A24}\ \ddot{R}_{\beta}^{i}. \tag{D-2}$$

The coefficient matrices in the above equations are available immediately following the flexible load operation for Body j. The quantities $\ddot{q}_{\ell}^{j}$ and $\ddot{\theta}_{\gamma}^{j}$ are elements of the dynamic derivative vector depicted as $\{\dot{X}\} = \{a\}$ in Figure 8.1 on page 90; thus, they are available immediately following a call to the derivative subroutine of the master program. The quantities $\dot{\omega}_{\beta}^{i}$ and $\ddot{R}_{\beta}^{i}$ are not available for any bodies other than for Body 1; consequently, their explicit calculation must be added to the program. From Equation (A-5):

$$\dot{\omega}_{\beta}^{j} = A_{\beta\alpha}^{ji}\ \dot{\omega}_{\alpha}^{i} + G_{\beta\gamma}^{j+}\ \ddot{\theta}_{\gamma}^{j} + S_{\beta}^{j5} \tag{D-3}$$

In a similar manner, Equation (A-32) can be rewritten to produce

$$\ddot{R}_{\alpha}^{j} = \ddot{R}_{\alpha}^{i} + L_{\alpha\gamma}^{i2}\ \ell_{\gamma}^{ij} - L_{\alpha\gamma}^{j2}\ \ell_{\gamma}^{ji} + L_{\alpha\gamma}^{i1}\ \dot{\ell}_{\gamma}^{ij} + A_{\alpha\gamma}^{ir(T)}\ \ddot{\ell}_{\gamma}^{ij}, \tag{D-4}$$

where

$$\begin{aligned} L_{\alpha\gamma}^{j1} &= A_{\alpha\beta}^{jr(T)}\ \tilde{\omega}_{\beta\gamma}^{j} - \tilde{\omega}_{\alpha\beta}^{r}\ A_{\beta\gamma}^{jr(T)} \\ L_{\alpha\gamma}^{j2} &= L_{\alpha\beta}^{j1}\ \tilde{\omega}_{\beta\gamma}^{j} - \tilde{\omega}_{\alpha\beta}^{r}\ L_{\beta\gamma}^{j1} + A_{\alpha\beta}^{jr(T)}\ \dot{\tilde{\omega}}_{\beta\gamma}^{j} - \dot{\tilde{\omega}}_{\alpha\beta}^{r}\ A_{\beta\gamma}^{jr(T)}. \end{aligned} \tag{D-5}$$

Thus, $\dot{\omega}_{\alpha}^{j}$ and $\ddot{R}_{\alpha}^{j}$ are computed recursively and can be considered as an addition to the rigid auxiliary calculations.

The above calculations for T_{α}^{ji} and F_{α}^{ji} are performed only at those time points called out for printing and/or plotting since the computations are only necessary for output.

D.1.2 Interconnection Loads for a Rigid Body

If Body j is a rigid body (with limb Body i), the interconnection forces and torques at its limb connection can also be determined using the equations from Appendix A. Specifically, from Equation (A-30) and the relationship $W_{\alpha}^{j\hat{s}} = F_{\alpha}^{ji}$, immediately preceeding Equation (A-33):

$$F_{\alpha}^{ji} = -C_{\alpha}^{A3} + B_{\alpha\ell}^{A31}\ \ddot{\theta}_{\ell}^{j\hat{s}} + B_{\alpha\beta}^{A32}\ \dot{\omega}_{\beta}^{j} + B_{\alpha\beta}^{A33}\ \ddot{R}_{\beta}^{j}. \tag{D-6}$$

Similarly, from Equations (A-30) and (A-37):

$$T_{\alpha}^{ji} = -\tilde{\ell}_{\alpha\beta}^{ji} F_{\beta}^{ji} - C_{\alpha}^{A2} + B_{\alpha\ell}^{A21} \ddot{\theta}_{\ell}^{j\hat{s}} + B_{\alpha\beta}^{A22} \dot{\omega}_{\beta}^{j} + B_{\alpha\beta}^{A23} \ddot{R}_{\beta}^{j} . \tag{D-7}$$

The coefficient matrices in the above equations are available immediately following the rigid load operation for Body j. The quantities $\ddot{\theta}^{j\hat{s}}$ are elements of the dynamic derivative vector, while $\dot{\omega}_{\beta}^{j}$ and $\ddot{R}_{\beta}^{j}$ are as given by Equations (D-3) and (D-4) with the single change of P_{α}^{j17} replacing S_{α}^{j5} in (D-3).

As in the previous case of a flexible body, the above calculations for T_{α}^{ji} and F_{α}^{ji} are performed only at those time points called out for printing and/or plotting.

D.2 Internal Load Calculations

In the UFSS program, flexible bodies are modeled in the traditional structural dynamics sense as a system of joints (or nodes) which are interconnected by weightless finite element members (e.g., beams, plates...). All masses are lumped at the joints. The orthogonal functions used to describe the spatial deformation of the bodies are normally taken to be the orthonormal modes produced by a standard structural dynamics program such as NASTRAN, SAMIS, or SMAP. In general, most such programs are based on small deflection theory, using the direct stiffness matrix finite element approach assuming linear stiffnesses. Such an approach allows for the generation of a specific transformation matrix, herein called the load transformation matrix (LTM), by a systematic application of unit forces along each degree of freedom with all other forces set equal to zero. In particular, such a matrix allows internal member forces to be calculated through the simple matrix multiplication operaton

$$L_M (t) = B_{MQ} F_Q (t). \tag{D-8}$$

In the above equation, each component of $P_M(t)$ represents a specific desired internal member load, B_{MQ} is the LTM and $F_Q(t)$ is the applied force vector. Specifically, B_{MQ} contains, as rows, coefficients for each degree of freedom relating the desired load to a unit force applied successively along each degree of freedom of the structure (assumed to be three times the number of joints in

UFSSP since only the modal translations are used to describe deformations), and the ordering of the columns must be identical to the degree of freedom ordering within the mode shapes. The applied force vector, $F_Q(t)$, contains both the externally applied forces (environmental forces, control forces...) and the inertial (d'Alembert) forces associated with the inertial velocity and acceleration of each joint.

Since the LTM is generated as a static problem with no relationship to the joint masses, the following development is intended to show how the inertial forces enter the calculations and exactly what model is to be used for generating the LTM.

D.2.1 Basic Concepts

Consider an arbitrarily moving flexible body, B, modeled in the traditional structural dynamics fashion as a system of N discrete joints, each having lumped mass m^n ($n = 1, \ldots, N$), with massless finite elements (beams, plates...) connecting the joints. As shown in Figure D.1, let $\underline{e}_\alpha$ ($\alpha = 1,2,3$) be unit vectors defining a right-handed, rectangular Cartesian reference frame fixed in the undeformed body (i.e., deformations of B are measured relative to to the $\underline{e}_\alpha$ frame). Let the origin of $\underline{e}_\alpha$ be denoted by 0, and let the external forces acting at 0 be denoted by $\bar{F}^o$. Let $\bar{a}^n$ be the inertial acceleration of joint n, and let $\bar{F}^n$ be the total external force acting on joint n. The governing equations of motion for joint n are as follows:

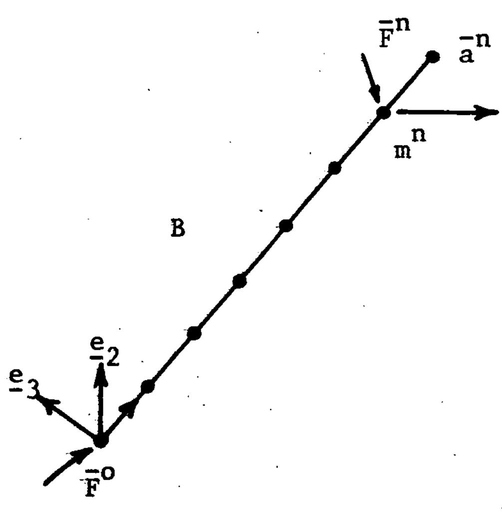


Figure D.1 - Schematic of a Flexible Body B

$$m^n \bar{a}^n = \bar{F}^n + \sum_{k=1}^{K} \bar{g}^{nk}, \tag{D-9}$$

where $\bar{g}^{nk}$ is the force exerted on joint n by the kth structural element (here, K is the total number of structural elements and, obviously, $\bar{g}^{nk}=0$ if element k does not connect to joint n).

Summing over the entire body B:

$$\sum_{n=1}^{N} m^n \bar{a}^n = \sum_{n=1}^{N} \left\{ \bar{F}^n + \sum_{k=1}^{K} \bar{g}^{nk} \right\} + \bar{F}^o$$

(note, if 0 is located at joint η of B, then conventionally $\bar{F}^\eta = 0$, as its contribution is expressed in $\bar{F}^o$)

But,

$$\sum_{n=1}^{N} \left(\sum_{k=1}^{K} \bar{g}^{nk} \right) = \sum_{k=1}^{K} \left(\sum_{n=1}^{N} \bar{g}^{nk} \right)$$

and, since the elements are massless,

$$\sum_{n=1}^{N} \bar{g}^{nk} \equiv 0.$$

Therefore,

$$\sum_{n=1}^{N} m^n \bar{a}^n = \sum_{n=1}^{N} \bar{F}^n + \bar{F}^o$$

or

$$\sum_{n=1}^{N} \left\{ \bar{F}^n - m^n \bar{a}^n \right\} + \bar{F}^o = 0 \tag{D-10}$$

at each instant of time.

Consider the same flexible body, B, rigidly cantilevered at 0, as shown in Figure D.2. Let $\bar{f}^n$ be the constant external force applied at joint n. Then, the static force equilibrium equations for B are as follows:

$$\sum_{n=1}^{N} \bar{f}^n + \bar{f}^o = 0.$$

where $\bar{f}^o$ is the reaction force vector acting on B at the cantilever point 0.

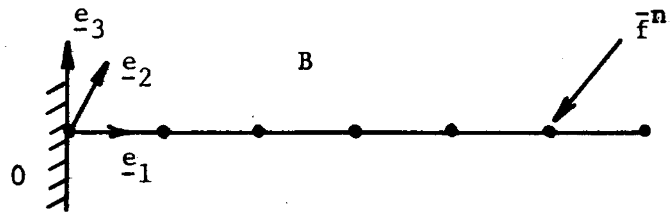


Figure D.2 - System B Cantilevered at 0

If, at any instant of time t, one now sets

$$\bar{f}^n = \bar{F}^n(t) - m^n\, \bar{a}^n(t), \tag{D-11}$$

then, necessarily,

$$\bar{f}^o = \bar{F}^o$$

and the internal loads within the cantilevered body shown in Figure D.2 are identical to those within the free body shown in Figure D.1 at time t.

One now generates the LTM for the system shown in Figure D.2 by successively applying a unit force along each translational degree of freedom with all other forces set equal to zero. Having obtained the LTM ($B_{\alpha\beta}$), one determines the desired internal loads within the system of Figure D.1 at any time t as being equal to those present in the system of Figure D.2 when the force distribution (D-11) is applied. Specifically, the internal loads $L_M(t)$ are determined as

$$L_M(t) = B_{MQ}\, F_Q(t), \tag{D-12}$$

where $F_Q(t)$ is a 3N-vector, with components

$$F_Q(t) = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ \cdot \\ \cdot \\ \cdot \\ f_3^N \end{Bmatrix}.$$

with f_{γ}^{n} being the component along the $\underline{e}_{\gamma}$ axis of the force distribution given by (D-11).

D.2.2 Internal Loads Program

Because of the core storage required to save the LTM and the fact that internal load calculations can be performed subsequent to the dynamic simulation of a given system, it is advisable that a separate progam "LØADS" be generated to calculate the internal loads within a given flexible body. Figure D.3 presents an overview of the program interfaces. As seen from Equation (D-12), only two basic quantities are required for the loads computation. The first quantity, B_{MQ}, is obtained from a structural dynamics program; the second quantity, $F_Q(t)$, is obtained from the UFSS program via a special history tape.

As previously mentioned, the row index, M, of B_{MQ} runs from one to A, where A is the total number of internal loads to be calculated; the column index, Q, runs from one to 3N, where N is the total number of joints in the structural dynamics model of the given flexible body, as input to UFSSP. Additional description of the LTM is contained in the user manual for the LØAD program referenced under "Associated Documentation" on page of this document.

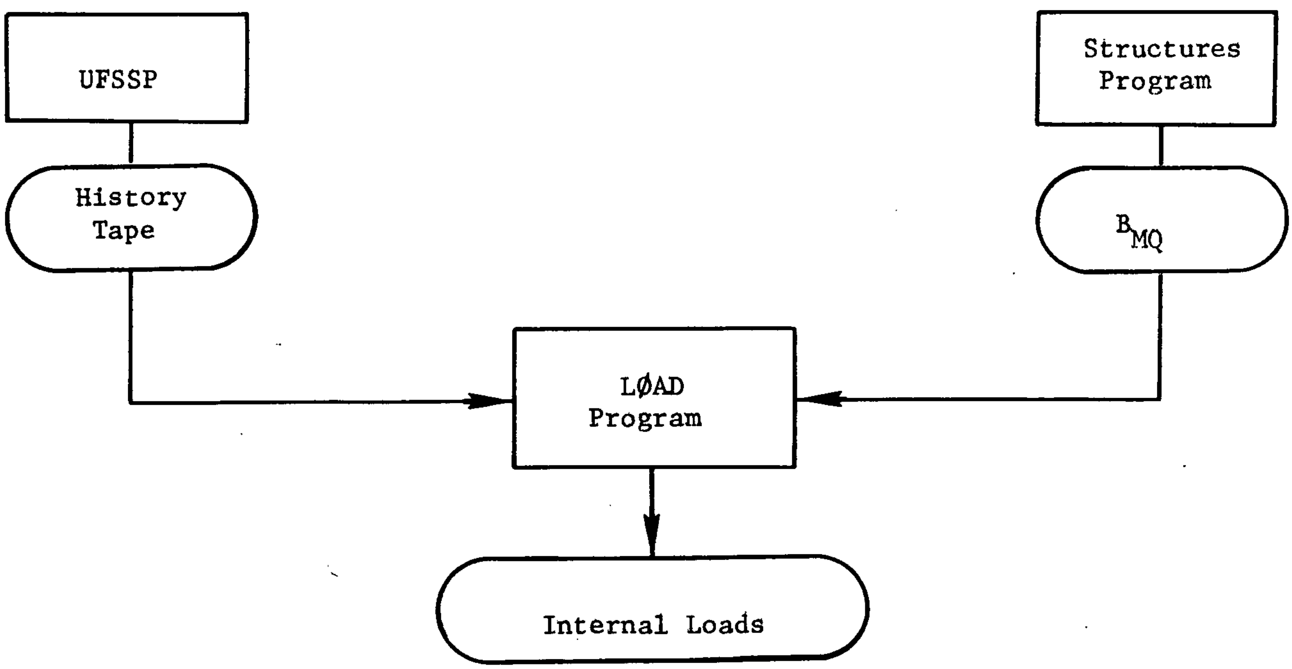


Figure D.3. Basic Interfaces for Internal Load Calculations

Thus, given Equation (D-12), it only remains to prescribe the computation of $F_Q(t)$; considering Equation (D-11), the problem is further reduced to determination of $\bar{f}^n$. Let us first determine $\bar{a}^n(t)$.

Differentiating Equation (A-2), one finds that for Body j:

$$\begin{aligned}\bar{a}^{jn} = \Big\{ & \ddot{\bar{R}}^r + \ddot{\bar{R}}^i + 2\left(\bar{\omega}^r \times \dot{\bar{R}}^i\right) + \left(\dot{\bar{\omega}}^r \times \bar{R}^i\right) \\ & + \bar{\omega}^r \times \left(\bar{\omega}^r \times \bar{R}^i\right) + \dot{\bar{\omega}}^i \times \bar{\ell}^{ij} + \bar{\omega}^i \times \left(\bar{\omega}^i \times \bar{\ell}^{ij}\right) \\ & + \ddot{\bar{\ell}}^{ij} + 2\left(\bar{\omega}^i \times \dot{\bar{\ell}}^{ij}\right)\Big\} \\ & + \dot{\bar{\omega}}^j \times \left(\bar{r}^{jn} + \bar{u}^{jn}\right) + \ddot{\bar{u}}^{jn} + 2\bar{\omega}^j \times \dot{\bar{u}}^{jn} + \bar{\omega}^j \times \left[\bar{\omega}^j \times \left(\bar{r}^{jn} + \bar{u}^{jn}\right)\right]\end{aligned}$$

or, specifically identifying that part of $\bar{a}^{jn}$ which is joint independent.

$$\begin{aligned}\bar{a}^{jn} = & \bar{A}^{Bj} + \dot{\bar{\omega}}^j \times \left(\bar{r}^{jn} + \bar{u}^{jn}\right) + \ddot{\bar{u}}^{jn} \\ & + 2\bar{\omega}^j \times \dot{\bar{u}}^{jn} + \bar{\omega}^j \times \left[\bar{\omega}^j \times \left(\bar{r}^{jn} + \bar{u}^{jn}\right)\right]\end{aligned}$$

where

$$\begin{aligned}\bar{A}^{Bj} = & \ddot{\bar{R}}^i + 2\left(\bar{\omega}^r \times \dot{\bar{R}}^i\right) + \left(\dot{\bar{\omega}}^r \times \bar{R}^i\right) + \bar{\omega}^r \times \left(\bar{\omega}^r \times \bar{R}^i\right) \\ & + \dot{\bar{\omega}}^i + \bar{\ell}^{ij} + \bar{\omega}^i \times \left(\bar{\omega}^i \times \bar{\ell}^{ij}\right) + \ddot{\bar{\ell}}^{ij} + 2\left(\bar{\omega}^i \times \dot{\bar{\ell}}^{ij}\right) \quad .\end{aligned}$$

In the above expression for $\bar{A}^{Bj}$, note the absence of the term $\ddot{\bar{R}}^r$. As in the case of the derivation of the basic dynamic equations in Appendix A (see, for example, the discussion leading to Equation (A-7)...), the d'Alembert force acting on Body j, if Body j were located at the origin of the reference frame, has been removed from the acceleration term by equating it to the corresponding gravitational force as reflected later in Equation (D-15).

Finally, in component form,

$$\begin{aligned}a_\alpha^{jn} = & A_\alpha^{Bj} + \ddot{W}_\alpha^{jn} + 2\,\tilde{\omega}_{\alpha\beta}^j\,\dot{W}_\beta^{jn} \\ & + \dot{\tilde{\omega}}_{\alpha\beta}^j\left(r_\beta^{jn} + W_\beta^{jn}\right) + \tilde{\omega}_{\alpha\beta}^j\,\tilde{\omega}_{\beta\delta}^j\left(r_\beta^{jn} + W_\beta^{jn}\right)\end{aligned} \tag{D-13}$$

where

$$W_\beta^{jn} = q_\ell^j \, \phi_{\ell\beta}^{jn}$$

$$\dot{W}_\beta^{jn} = \dot{q}_\ell^j \, \phi_{\ell\beta}^{jn}$$

$$\ddot{W}_\beta^{jn} = \ddot{q}_\ell^j \, \phi_{\ell\beta}^{jn} \quad .$$

The external forces acting on joint n ($\bar{F}^{jn}$) will presently include gravitational forces and control thruster forces. (Geomagnetic, solar pressure and aerodynamic pressure forces as well as control torquers can be added at any time if a specific application calls for their inclusion.) From Equation (8-16), the control thruster force components at joint n are given by

$$F_\alpha^{jnC} = A_{\alpha\beta}^{jr} \, R_\beta^{jnT} \quad . \tag{D-14}$$

From Equation (B-11) and the fact that $\underline{a}^r$ has components $-A_{\alpha 3}^{jr}$ in the $\underline{e}_\beta^j$ frame and components { 0 0 -1 } in the $\underline{e}_\beta^r$ frame, the gravity force components at joint n are given by

$$\begin{aligned} F_\alpha^{jnG} = &- \frac{\gamma m^{jn}}{(R^r)^3} \left[A_{\alpha\beta}^{jr} \, R_\beta^j + (r_\alpha^{jn} + W_\alpha^{jn}) \right] \\ &+ \frac{3\gamma m^{jn}}{(R^r)^3} \left[R_3^j + A_{3\alpha}^{jr\,(T)} \, (r_\alpha^{jn} + W_\alpha^{jn}) \right] A_{\alpha 3}^{jr} \quad . \end{aligned} \tag{D-15}$$

(Note absence of the term $\frac{-\gamma m^{jn} \bar{R}^r}{(R^r)^3}$ which has been equated to the term $m^{jn}\ddot{R}^r$ arising from $\bar{A}^{Bj}$ as previously noted.)

Finally, the external force acting on joint n is given by

$$F_\alpha^{jn} = F_\alpha^{jnC} + F_\alpha^{jnG} \tag{D-16}$$

and the total applied force components making up $F_Q(t)$ for use in Equation (D-12) are given by

$$f_\alpha^{jn} = F_\alpha^{jn} - m^{jn} \, a_\alpha^{jn} \tag{D-17}$$

with F_α^{jn} and a_α^{jn} as given by Equations (D-16) and (D-13) respectively.

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